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Theoretical foundations of the chronometric cosmology

(redshift/chronometric theory)

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ABSTRACT The derivation of the redshift (z)–distance (r) relation in the chronometric theory of the Cosmos is amplified. The basic physical quantities are represented by precisely defined self-adjoint operators in global Hilbert spaces. Computations yielding explicit bounds for the deviation of the theoretical prediction from the relation $z = \tan^2(r/2R)$ (where R denotes the radius of the universe), earlier derived employing less formal procedures, are carried out for: (*a*) a cut-off plane wave in two dimensions; (*b*) a scalar spherical wave in four dimensions; (*c*) the same as (*b*) with appropriate incorporation of the photon spin. Both this deviation and the (quantum) dispersion in redshift are shown to be unobservably small. A parallel classical treatment is possible and leads to similar results.

1. Introduction. The chronometric theory (1–3) is a modification of special relativity whose salient features are: (*a*) globally, space is spherical, although locally space-time is Minkowskian; (*b*) the physical time (or dually, energy) differs globally from that of special relativity, although for localized states the difference is unobservably small; (*c*) technically, the theory is based on considerations of symmetry, and in particular group-theoretic properties of the Maxwell equations; Riemannian geometry is only incidental. Its chief and definitive novelty is (*b*), which directly implies a loss of special relativistic energy for particles freely propagated over large distances, and does so without infringement of Lorentz invariance or conservation of the total energy. Philosophically, (*b*) represents a continuation of the nonanthropocentric tradition, in that it distinguishes between the observed time x_0, which takes the same form as in special relativity, and global physical time t, which is synchronous with x_0 in the short run within terms of third or higher order. The impossibility of direct observation of t—indirectly, it may be observed via the redshift—is partially in the direction of the indeterminacy principle, but differs in part in that it represents a limitation of human scale and facilities, rather than an inherent limitation of nature.

While these are substantial modifications, the theory is in a way qualitatively quite limited, being primarily kinematical. There is no direct dynamical content, i.e., statement as to the forms of the interactions between free systems, apart from the implication of a slightly modified symmetry group. This, however, deforms into (or has as a "limiting case," in Minkowski's term, or "contracts to," in Wigner's term) the symmetry group of special relativity, i.e., the inhomogeneous Lorentz group augmented by scale transformations, as the radius R of the spherical space component of the chronometric cosmos becomes infinite. As a theory of free systems in reference space, it appears to meet all physically indicated general requirements of temporal and spatial homogeneity and isotropy: Lorentz invariance, causality and finiteness of propagation velocity, and positivity of the energy. It provides thereby a possible basis for quantum field theory which appears conceptually as appropriate as special relativity.

From the standpoint of general relativity, the chronometric cosmos provides a proper model for empty reference space by virtue of its conformal flatness. From the standpoint of group-invariance the theory is terminal in that its symmetry group cannot be obtained by the deformation of any other symmetry group of the same dimension (ref. 4, p. 264). Indeed, the theory was originally developed for elementary particle applications, and it enjoys in particular the possibility of discrete mass multiplets of the type which are forbidden by the O'Raifeartaigh theorem (5, 6) for a broad class of models based on special relativity.* From a general standpoint, an interesting feature is that the physical interpretations of the generators of the group as energy, angular momenta, etc., are completely fixed by the algebra of the group, as are fundamental physical units by choosing the non-vanishing structure constants of the group to have the values ± 1. These units may otherwise be specified by setting $\hbar = c = R = 1$, where R is the radius of space, as is done throughout this note.

The chronometric redshift arises not as a dynamical effect, but solely from the phenomenological postulate that the wavelength of photons as measured is inherently scale-covariant. That is to say, if the conventional standard of length is diminished by the factor λ, then the measured wavelength is diminished by the same factor, not merely approximately, but in principle completely exactly, by virtue of the nature and limitations of the measuring process. But in a symmetrical curved universe, such as the chronometric cosmos, there is no *a priori* reason for the energy (or corresponding notions of wavelength or time) to be scale-covariant in this sense. The radius of the space component of the chronometric cosmos $\tilde{M}$ provides in fact a distinguished distance scale, albeit one not directly accessible to a human observer. While global changes of scale are applicable in $\tilde{M}$, the natural energy and time in $\tilde{M}$ are not correspondingly scale-covariant. Moreover, it is only in the small, and there only within terms of order R^{-1}, that a change of scale leaves

* The treatment of the scalar case in (7) is readily extended to fields of higher spin. Fields representing particles of a given mass $m > 0$ are obtained by restriction of the full symmetry group $\tilde{SO}(4,2)$ of the chronometric theory, in its natural action on scalar, spinor, and other fields, to the subgroup $\tilde{SO}(3,2)$ generated by the chronometric energy and homogeneous Lorentz transformations. More specifically, such particles are represented by vectors in the eigenspace of the Casimir operator C of $\tilde{SO}(3,2)$ with eigenvalue m^2. As the radius R of the universe tends to ∞, $\tilde{SO}(3,2)$ deforms into the inhomogeneous Lorentz group, while the equation $C\psi = m^2\psi$ deforms into the Klein-Gordon, Dirac, or other relativistic wave equation. The admissible mass spectrum is further restricted by physical requirements (unitarity, microcausality, the existence of nontrivial local couplings, etc.) whose implications pose well-defined mathematical problems. In particular, weak decays appear connected with non-unitary representations having unitary composition series.

invariant the observational separation of space-time into time and space components.

The question of the precise theoretical description of what is actually measured in a laboratory determination of wavelength might appear ambiguous, were it not for the mathematical fact that, apart from the chronometric time (or energy), there is locally in Minkowski space only one other possible formulation of time (or energy) which meets very general and natural requirements of causality and symmetry, including Lorentz-invariance: namely that of special relativity. From the chronometric standpoint, this *laboratory-measured* energy may be characterized as the scale-covariant component of the true *physical driving energy*. More specifically, in the chronometric cosmos, the total energy H can be uniquely expressed in the form $H = H_0 + H_1$, where the H_i ($i = 0,1$) are respectively scale- and anti-scale-covariant, in the sense that $[H_i,K] = (-1)^i H_i$, where K is the infinitesimal generator of the one-parameter group of scale transformations. In view of the Lorentz-invariance established by elementary particle experiments, and the intrinsic scale covariance of the laboratory measurement process, it follows that a laboratory determination of energy can only be appropriately represented by H_0. The chronometric theory thus provides an explanation for the redshift which is natural in that the redness is automatic rather than assumed in order to conform to observation as in Friedmann-Robertson models, and which, unlike these models, has the properties of Lorentz invariance or global conservation of energy. The theory is mathematically unique[†] within its genre, assuming that the physical Cosmos is four-dimensional and endowed with a Lorentzian notion of causality; it involves no free parameters, apart from the distance scale set by the value R of the radius of the universe in conventional units. It predicts relations between the luminosities, redshifts, angular diameters, and numbers of extragalactic objects which are independent of R, or of other adjustable parameters, but are nonetheless in excellent agreement with the observed relations for systematic samples of such objects. In contrast, the general-relativistic expanding-universe model predicts relations dependent on two parameters, q_0 and Λ; but there are no values of these parameters for which the naive predictions of this model are in good agreement with the data for large or otherwise statistically cogent (e.g., complete) samples of galaxies, quasars, or radio sources. To be sure, satisfactory agreement may be obtainable by postulating a variety of effects dependent on redshift (such as "evolution"); but these are derogatory to the general appeal and scientific economy of the theory, as well as quite disruptive of its predictive power.

It is not the purpose of this note to treat the observational validation of the chronometric theory or the limitations of the expansion theory; these matters are detailed in refs. 3, 8[‡], and 9, and elsewhere. Having clarified in general terms the nature and relevance of the chronometric theory, the present note aims to elaborate its analytical foundations, in the form of a global and rigorous treatment of its redshift-distance relation. This is not in itself an observed relation, but only elementary geometry is involved in the deduction from this theoretical relation of the predicted relations between the observed quantities cited. Earlier, technically relatively simple deductions of the redshift-distance law have been indicated in refs. 1–3; in these treatments, a photon of wave function ψ was regarded as having apparent frequency ν near a point P if $-i(\partial/\partial t)\psi \sim \nu\psi$, near P. But, rigorously speaking, physically realizable photons must have finite energy and spatial extent; they are not precisely plane waves; their wave functions ψ lie in a unique Lorentz-invariant Hilbert space, and the apparent frequency is properly given by the standard principles of quantum mechanics as the expectation value of the energy operator in the state ψ, i.e., $\langle -i(\partial/\partial t)\psi,\psi\rangle$, assuming that ψ is normalized to unit norm. This expectation value, and the corresponding dispersion, involve integrals over all of space. While it is physically plausible that these may be approximated by cutoff integrals restricted to regions contained in neighborhoods of the points of emission and observation, the crucial importance of the redshift-distance relation for the observational validation of the theory, together with uncertainty as to the validity of the plane wave representation for analysis of propagation through very large distances, indicates importance for the rigorous confirmation of the validity of the cutoffs by treatment of the exact Hilbert space inner products. The present note details this treatment for representative wave functions of increasing complexity, and indicates a corresponding classical derivation.

2. The Photon Hilbert Space. With the scalar approximation generally employed in redshift considerations, a photon may be represented by a solution φ of the wave equation $\Box\varphi = 0$, where $\Box = (\partial/\partial x_0)^2 - (\partial/\partial x_1)^2 - \ldots - (\partial/\partial x_n)^2$, n being the number of space dimensions. Upon emission, the photon wave function is localized in a small region of space; mathematically its wave function φ vanishes outside this region, at the time in question[§]. Such a wave function takes the form $\varphi(X) = (2\pi)^{-n/2}\int \exp[iX\cdot K]F(K)dK$, where F is a distribution in the sense of L. Schwartz, $K = (k_0,k_1,\ldots,k_n)$, $X = (x_0,x_1,\ldots,x_n)$, $X\cdot K = x_0k_0 - x_1k_1 - \ldots - x_nk_n$, and $dK = dk_0dk_1\ldots dk_n$; and

$$F(K)dK = [\delta(k_0 - |k|)f_+(k) + \delta(k_0 + |k|)f_-(k)]|k|^{-1}dk, \quad [2.1]$$

where $k = (k_1,\ldots,k_n)$, $dk = dk_1\ldots dk_n$, and $|k| = (k_1^2 + \ldots + k_n^2)^{1/2}$. The wave function φ is normalizable in case $\int |f_\pm(k)|^2|k|^{-1}dk < \infty$, where $dk = dk_1\ldots dk_n$; cf. ref. 10. The reality of φ is equivalent to the hermitian character of F, i.e., $F(-K) = \bar{F}(K)$, which in turn is equivalent to the condition that $f_-(k) = \bar{f}_+(-k)$. In this case, setting $f = f_+$, $f(k)$ may be expressed in terms of the Cauchy data at time 0 as

$$f(k) = 2^{-1}(2\pi)^{-n/2}[|k|\int e^{is\cdot k}\varphi(x,0)dx - i\int e^{+ix\cdot k}\dot{\varphi}(x,0)dx], \quad [2.4]$$

where $x\cdot k = x_1k_1 + \ldots + x_nk_n$ and $dx = dx_1\ldots dx_n$.

The scalar photon Hilbert space H may be defined as consisting of all real normalizable solutions of the wave equation, with the inner product $\langle\varphi_1,\varphi_2\rangle = \int f_1(k)\bar{f}_2(k)|k|^{-1}dk$, where f_j is the momentum-space function corresponding to φ_j ($j = 1,2$), and where the action of the complex unit i is defined by its normal action on the corresponding analytic signal. The orthochronous conformal groups acts in H by unitary transformations, which define thereby a continuous

† More specifically, there is only one type of local observational framework in Minkowski space other than the conventional relativistic type, which has intuitive and empirically indicated properties of temporal and spatial homogeneity and isotropy; any two frameworks of the same type are connected by a local causality-preserving transformation (cf. ref. 3).

‡ In Eq. 4 of ref. 8, 2.5 should be replaced by −2.5.

§ This is the physical, mathematically real, wave function, which is more convenient here than the corresponding positive-frequency component, or analytic signal, which vanishes almost nowhere. Cf., e.g., Born and Wolf (13).

projective representation of the conformal group which extends the more familiar representation of the inhomogeneous Lorentz subgroup (n being odd); and the same is true of the solution manifolds of the Maxwell and other cited equations. In the case of the scalar wave equation, this action is not simply scalar transformation, but involves additional suitable multiplications, except in the special case $n = 1$; cf. ref. 12.

The Maxwell field is definable in terms of the second-order form

$$\omega = E_1 dx_2 dx_3 + E_2 dx_3 dx_1 + E_3 dx_1 dx_2 + B_1 dx_0 dx_1 + B_2 dx_0 dx_2 + B_3 dx_0 dx_3,$$

where (E_1,E_2,E_3) and (B_1,B_2,B_3) are the usual electric and magnetic vectors. Maxwell's equations take the form $d\omega = \delta\omega = 0$, where d denotes the usual differentiation operator on forms and δ is its adjoint d^* with respect to the adjoint operator $*$ on forms, corresponding to the Minkowski metric. The conformally invariant inner product in the form given in ref. 10 is:

$$\langle \omega,\omega' \rangle = \int [\hat{E}_1(k)\bar{\hat{E}}_1'(k) + \ldots + \hat{B}_3(k)\bar{\hat{B}}_3'(k)]|k|^{-3}dk,$$

where the superscribed carets denote Fourier transforms.

3. Redshift Analysis for Conformally Invariant Systems. Let U denote a unitary representation of the conformal group G, or any covering group thereof, in the Hilbert space K, which represents a conformally invariant physical system, such as the Hilbert space H indicated in Sect. 2. Among the generators of the group G are the two distinguished ones associated with the natural chronometric time t and the special relativistic time x_0; a change of Lorentz frame or scale alters the analytic form of these times, but does not affect the relations within the group G of the corresponding generators $\partial/\partial t$ and $\partial/\partial x_0$, apart from a physically unobservable unitary equivalence. According to the chronometric theory, the physical (driving) hamiltonian is the operator $H = -iU(\partial/\partial t)$ corresponding to the advance of chronometric time, while direct laboratory observations of the energy yield only the scale-covariant component $H_0 = -iU(\partial/\partial x_0)$ of H. This component does not commute with H and so is not conserved; after an elapsed chronometric time s, it is represented, in the Heisenberg picture, by the operator $H_0(s) = e^{-isH}H_0 e^{isH}$. It is a matter of pure group theory that

$$H_0(s) = \alpha H_0 + \beta H_1 + \gamma K',$$

where $H_1 = H - H_0$ and $K' = 2i[H_0,H_1]$ (we also use $K = iK'$), while

$$\alpha = (1 + \cos s)/2, \beta = (1 - \cos s)/2, \text{ and } \gamma = \sin s/2.$$

The redshift z is defined so that $1 + z$ is the factor by which the special relativistic energy is reduced, in the state in question; i.e., $\langle H_0(s)\rangle = (1 + z)^{-1}\langle H_0\rangle$. In order to determine z it therefore suffices to evaluate $\langle H_0\rangle$, $\langle H_1\rangle$ and $\langle K'\rangle$.

Redshift observations are basically of quasi-monochromatic sources. After redshifting the linewidth remains small compared to the wavelength. Any quantum theory of the redshift is constrained to show that the quantum dispersion is too slight to affect the observed linewidth. The dispersion in observed frequency is here $[\langle H_0(s)^2\rangle - \langle H_0(s)\rangle^2]^{1/2}$; and it is evident that this is readily determined from the expectation values of the squares and products of H_0, H_1, and K'. The Schwarz inequality implies the following explicit estimates for the deviation of the observed frequency from that given by the simple law $z = \tan^2(s/2)$ and for the (quantum) variance in the observed frequency:

$$|\langle H_0(s)\rangle - (1/2)(1 + \cos s)\langle H_0\rangle| \leq \langle H_1^2\rangle^{1/2} + \frac{1}{2}\langle K'^2\rangle^{1/2};$$

$$\langle H_0(s)^2\rangle - \langle H_0(s)\rangle^2 \leq \alpha^2[\langle H_0^2\rangle - \langle H_0^2\rangle] + \langle H_1^2\rangle + \frac{1}{4}\langle K'^2\rangle + 2\langle H_1^2\rangle^{1/2}\langle H_0^2\rangle^{1/2} + 4\langle K'^2\rangle^{1/2}[\langle H_0^2\rangle^{1/2} + \langle H_1^2\rangle^{1/2}].$$

The redshift analysis may equally well be conducted in the Schrödinger picture, by standard quantum mechanics, and in this form is more readily adapted to a classical treatment. If the photon is emitted in the state ψ, after time s it is, according to the chronometric theory, in the state $e^{isH}\psi = \psi_s$, say, and z may be defined by the condition that $1 + z$ is the factor by which the expectation value of H_0 in the state ψ_s is less than its original expectation value, in the state ψ. This is analytically equivalent to the computation in the Heisenberg representation, as is also the computation of the quantum dispersion in the observed energy.

In the classical treatment, the physical energy E is a functional rather than an operator, but remains the sum of unique scale- and anti-scale-covariant components E_0 and E_1. For a localized state, E differs negligibly from the observed relativistic energy E_0, but as the state becomes delocalized in the course of temporal evolution E_0 may diminish. Classically, the redshift z after elapsed time s is given by the equation: $E_0(s) = (1 + z)^{-1}E_0(0)$, where $E_0(s)$ denotes the total relativistic energy of the wave at time s. As in the case of the quantum hamiltonians, the super-relativistic energy E_1 is the transform of the relativistic energy by conformal inversion, as shown by a Poisson bracket analysis analogous to that involving operator commutators for the quantum case. In, e.g., the case of the wave equation in flat space, $\Box\varphi = 0$, E_0 takes the form: $E_0 = \int[(\text{grad}\varphi)^2 + \dot{\varphi}^2]dx$, and a simple computation shows that in terms of the field at time 0 (but not at other times)

$$E_1 = (1/4)\int[(\text{grad}\,\varphi)^2 + \dot{\varphi}^2]x^2dx.$$

4. Cut-Off Plane Waves. A plane wave $\varphi(X) = e^{i\nu(x_0 - k\cdot x)}$ is not normalizable, nor does it become so when cut off spatially in four space-time dimensions. However, on choosing an axis along the direction of propagation, $\varphi(X)$ may be represented in the 2-dimensional form $e^{i\nu(x_0 - x_1)}$, which becomes normalizable when spatially cut off.

Consider, therefore, a 2-dimensional wave emitted with wave function $\varphi(x_1,x_0) = g(x_1 - x_0)$, where g is a function of a single variable to be specified later. The basic operators involved in the evaluation of the redshift are, as conformal vector fields in 2-dimensional Minkowski space M: $iH_0 = \partial/\partial x_0$: $iH_1 = (1/4)(x_0^2 + x_1^2)(\partial/\partial x_0) + (1/2)x_0x_1(\partial/\partial x_1)$;

$$K = x_1(\partial/\partial x_1) + x_0(\partial/\partial x_0).$$

Corresponding actions on g are therefore:

$$H_0: g(x) \longrightarrow ig'(x); H_1: g(x) \longrightarrow (1/4)x^2g'(x); K: g(x) \longrightarrow xg'(x).$$

For wave functions of the form $\varphi_j(x_1,x_0) = g_j(x_1 - x_0)$ ($j = 1,2$), the inner product takes the form $\langle\varphi_1,\varphi_2\rangle = \int_{-\infty}^{\infty} \hat{g}_1(k)\hat{g}_2(k)\, kdk$. It follows that if in particular g takes the form $g(y) = G(\nu y)$, where G is a given function independent of the emitted frequency ν, the required inner products depend on ν in the following way: $\langle H_0^2\varphi,\varphi\rangle \propto \nu^2 \langle H_0\varphi,\varphi\rangle$,

$\langle H_0 K\varphi,\varphi\rangle \propto \nu$; $\langle\varphi,\varphi\rangle$, $\langle K\varphi,\varphi\rangle$, $\langle K^2\varphi,\varphi\rangle \propto \nu^0$; $\langle H_1\varphi,\varphi\rangle$, $\langle H_1 K\varphi,\varphi\rangle \propto \nu^{-1}$; $\langle H_1{}^2\varphi,\varphi\rangle \propto \nu^{-2}$. It suffices, therefore, to evaluate the inner products when $\nu = 1$, and setting $h = \hat{G}$, and assuming for simplicity henceforth that h is real, the required inner products are readily expressed in terms of low-order moments on the interval $0 < x < \infty$ of the functions $xh(x)^2$, $x[xh(x)]'^2$, and $[xh(x)]''^2$.

Now specializing to the case of a cut-off plane wave, G may be taken to have the form: $G(y) = 1 + \cos y$ when $|y| \leq p$, where $p = n\pi$, n being an odd integer; and $G(y) = 0$ otherwise; neglecting an irrelevant constant factor, we take consequently $h(x) = \sin px/x(x^2 - 1)$. The resulting integrals may be appropriately bounded by separate estimations for the regions $x^2 \ll 1$, $x^2 \sim 1$, and $x^2 \gg 1$. When the cut-off wave includes at least one full cycle, i.e., $n \geq 1$, it is found that $\langle\varphi,\varphi\rangle \geq \pi n$; $\langle K\varphi,K\varphi\rangle/\langle\varphi,\varphi\rangle \leq 53n^2$; $\langle H_1\varphi,H_1\varphi\rangle/\langle\varphi,\varphi\rangle \leq 686n^4$. Substituting in the inequalities given earlier, it follows that, in chronometric units:

$$|\langle H_0(s)\rangle - (1/2)(1 + \cos s)\langle H_0\rangle| \leq 27n^2\nu^{-1} + 4n \leq (27r^2 + 4r)\nu.$$

where $r = n\nu^{-1}$ is the radius of the region outside of which the emitted wave vanishes. Thus the deviation in the observed frequency $\langle H_0(s)\rangle$ from the value given by the general formula $z = \tan^2(s/2)$ is, relative to the emitted frequency, at most $27r^2 + 4r$. Even if the radius r of the region within which the original emitted wave was localized were of the apparent order of magnitude of the core of a Seyfert galaxy, say ~1 pc, this deviation would be unobservably small. For the radius R of the chronometric universe is $\gtrsim$100 Mpc, on the basis of Virgo observations and the redshift law $z = \tan^2(d/2R)$, where d is the distance to the galaxy observed. It results that $r \leq 10^{-8}$, and follows in turn that a quite conservative bound for the deviation of the expected redshift from the general law $z = \tan^2(d/2R)$ is at most 1 part in 10^7. This is negligible relative to the intrinsic dispersion arising from typical motions within the source, etc.

Similar estimates apply to the expected (quantum) dispersion in redshift; specifically, if ν denotes the emitted frequency, in which there is dispersion σ, and ν' denotes the observed frequency, of quantum dispersion σ', then

$$(\sigma'/\nu')^2 \leq (\sigma/\nu)^2 + 1.2 \times 10^{-8}(1 + z)^2.$$

The quantum line broadening $\Delta\lambda/\lambda$ is thus at most $1.1 \times 10^{-4}(1 + z)^2$, or less than the intrinsic observational dispersion σ/ν, for presently observable redshifts. It should perhaps be noted that these deviations represent mathematical limits, and that the true physical orders of magnitude may well be some orders of magnitude smaller.

For a classical treatment of the same wave functions, it is convenient to use the chronometric time and space coordinates τ and ρ, defined by the equations:

$$\tan \tau = x_0(1 - X^2/4)^{-1}; \tan \rho = x_1(1 + X^2/4)^{-1}.$$

In terms of these variables, $\varphi(x_1,x_0) = g[2\nu \tan(\rho - \tau)/2]$, and φ is given after the elapsed time s by the substitution of $\tau + s$ for τ. In this way it follows that

$$E_0(\varphi_s) = 2\nu^2 \int g'[2\nu \tan(\rho - s)/2]^2 \cos^2(s/2) \times \sec^4[(\rho - s)/2]d\rho.$$

If g is localized near 0, the integrand vanishes except for $\rho \sim s$, and follows that $E_0(\varphi_s)/E_0(\varphi) \sim \cos^2(s/2)$, which gives the relation $z = \tan^2(s/2)$ obtained earlier.

From the classical standpoint, the redshift takes place through the delocalization of the emitted wave, which results in the nontriviality of the super-relativistic energy component E_1, which initially is $\lesssim(p/\nu)^2$, and is hence unobservably small. The conservation law for the total energy $E_0 + E_1$ becomes manifest when the wave equation is expressed in the equivalent form $\partial^2\varphi/\partial\tau^2 - \partial^2\varphi/\partial\rho^2 = 0$, from which it is evident that $\int[(\partial\varphi/\partial\rho)^2 + (\partial\varphi/\partial\tau)^2]d\rho$ is conserved; the last expression is an alternative form for the total chronometric energy, consituting the classical analogue to $H_0 + H_1$ as in the operator representing $-i\partial/\partial\tau$, in that it gives the infinitesimal temporal development (in terms of τ) as a Poisson bracket.

5. Spherical Waves in Four Dimensions with Gaussian Spatial Cut-Off. In four dimensions, the spherical wave

$$\varphi(x, x_0) = \int_{k^2=1} \cos \nu x_0 \cos \nu k\cdot x d\mu(k) = \cos \nu x_0 \frac{\sin \nu r}{\nu r}$$

where μ denotes the uniform probability measure on the sphere $k^2 = 1$, is not normalizable, but becomes so on replacing it by the wave function φ_a whose Cauchy data at the time $x_0 = 0$ are obtained from those of φ by multiplication with $e^{-(1/2)(x^2/a^2)}$. Since $\dot{\varphi}(x,0) = 0$, it follows from the earlier analysis that the function $f_a(k)$ associated with φ_a has the form

$$2f_a(k) = (2\pi)^{-3/2}|k| \int e^{ix\cdot k} \frac{\sin |x|}{|x|} e^{-|x|^2/2a^2} dx$$

which can be evaluated as

$$f_a(k) = (a/4\nu)e^{-(1/2)a^2(\nu^2+|k|^2)}[e^{a^2\nu|k|} - e^{-a^2\nu|k|}].$$

The actions of H_0, H_1, and K in terms of the function f_a are, for general f:

$$H_0: f(k) \longrightarrow |k|f(k); H_1: f(k) \longrightarrow -(|k|/4)\Delta f;$$

$$K: f(k) \longrightarrow -\sum_{j=1}^{3}(\partial/\partial k_j)(k_j f).$$

Introducing the parameter $n = b\nu$ analogous to that employed earlier and simplifying, expressions for the inner products are obtained which apart from simple factors depend on integrals of the form $\int_{\pm n}^{\infty} s^b e^{-s^2} ds$, or slight variants thereof, for $b = 0,1,\ldots,7$. These are readily estimated, with the (crude but adequate) results, for arbitrary $n \geq 1$:

$$\langle\varphi,\varphi\rangle \geq 4^{-1}\pi^{3/2}n; \langle K'^2\rangle \leq 3 + 23n^2; \langle H_1{}^2\rangle \leq 25a^2n^2.$$

The bounds on $\langle K'^2\rangle$ and $\langle H_1{}^2\rangle$ are of the same general form, but somewhat smaller, than those in the preceding section, and lead to the same conclusion: The chronometric prediction is for a redshift–distance law which is independent of the photon wave function (or frequency), within the limits of feasible observation, and given by the equation $z = \tan^2(r/2R)$; in addition to which, spectral lines remain sharp after redshifting.

6. Large-Distance Effect of Spin-Orbit Coupling. In the case of the full Maxwell equations only the inner products involving H_1 are not deducible from the earlier scalar results. The action of iH_1 is to transform ω into the form ω' whose coefficients are in vector form:

$$\mathbf{E}' = Y\mathbf{E} + x_0\mathbf{E} + (1/2)\mathbf{x} \times \mathbf{B},$$
$$\mathbf{B}' = Y\mathbf{B} + x_0\mathbf{B} + (1/2)\mathbf{x} \times \mathbf{E},$$

where Y denotes the scalar vector field

$$-(1/4)X^2(\partial/\partial x_0) + (1/2)x_0 K.$$

In order to show that the photon spin has no significant effect on the redshift, it suffices by the Schwarz and Minkowski inequalities to show that $\|H_1\omega\| \ll \nu\|\omega\|$. The emitted photon wave function will be assumed to be the minimal spin 1 modification of the scalar one earlier employed in 4-dimensional space, and we consequently take for the Fourier transforms of the Cauchy data at time 0:

$$\hat{E}_1(k) = k_2|k|f \quad \hat{E}_2(k) = -k_1|k|f \quad E_3 = 0$$

$$\hat{B}_1(k) = (k_1k_3)f, \quad \hat{B}_2(k) = (k_2k_3)f, \quad \hat{B}_3(k) = -(k_1^2 + k_2^2)f$$

The $YE_1, \ldots, YB_3$ components of the coefficients of ω' are the same as the scalar terms already treated, except that f is replaced by its multiple by a simple expression which is homogeneous in the k_j, of first degree. Consequently the bounds for these components are essentially the same as in the scalar case. The novelty lies only in the need to estimate terms of the form $(1/2)(B_3x_2 - B_2x_3)$, and the like; the terms similar to E_1x_0 vanish at $x_0 = 0$ and so do not contribute.

Computation shows that each of the $\hat{E}_i(k)$ and $\hat{B}_i(k)$ is bounded by $3|k|p(|k|) + |k|^2p'(|k|)$. Using the Minkowski inequality it follows that

$$\|H_1\omega\| \leq \text{orbital term} + 12\pi^{1/2}\left[3\left(\int_0^\infty xp(x)^2dx\right)^{1/2} + \left(\int_0^\infty x^3p'(x)^2dx\right)^{1/2}\right].$$

On the other hand, $\|\omega\|^2 = (32\pi/3)\int_0^\infty xp(x)^2dx$. By procedures similar to those earlier indicated, it follows that the spin contribution to the bound for $\langle H_1^2\rangle^{1/2}$ is at most $1 + 2an$ for $n \geq 1$, and so is smaller than the orbital bound.

7. Comment. Our main results, consisting of the relation $\langle H_0(s)\rangle \sim [(1 + \cos s)/2]\langle H_0(0)\rangle$ and the estimate $\sigma_{\nu_{obs}}/\nu_{obs} \lesssim \sigma_{\nu_{em}}/\nu_{em}$, appear to be independent of the character of the cut-off, within wide limits, and it is only in order to obtain rigorous explicit bounds that specific cut-offs appear necessary.

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