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SHORT COMMUNICATIONS

Application of Lobachevsky's Formula on the Angle of Parallelism to Geometry of Space and to the Cosmological Redshift

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Abstract. A new formula for an experimental determination of a fundamental quantity in hyperbolic geometry, namely, Lobachevsky's $\pi(\delta)$ function is presented. Applications to cosmology are discussed. Using the above result, we strongly suggest that the sign of the curvature of the ambient space is negative.

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1. INTRODUCTION

We present a new formula (7) which enables us to experimentally find Lobachevsky's function $\pi(\delta)$. This brings us closer to the verification of some speculations on the geometry of physical space.

Due to the negative curvature of Lobachevsky space, the notion of length is coupled to the notion of angle via the so-called *angle of parallelism* π versus the distance δ, $\pi = \pi(\delta)$. In Euclidean geometry, angle and length are independent notions. The duality between length and angle in Lobachevsky geometry is usually expressed as a property of nonexistence of unequal similar triangles in Lobachevsky geometry or, equivalently, the dependence of the sides of a triangle on the angles.

The significance of the function $\pi(\delta)$ is that this function is a direct (primary) indicator of the negative curvature of space. Thus, the ability to find $\pi(\delta)$ in terms of experimental data, as we present it, is extremely important. It is easy to see that if $\pi(\delta) = 90^\circ$ (see (3)), then our space is Euclidean. Thus, an experimental determination of $\pi(\delta)$ is that of the actual curvature of space.

In the present paper, we do not discuss Lobachevsky geometry by itself and refer the reader to the references. In particular, Anderson [1] provides an overall modern exposition of hyperbolic geometry at an introductory level. The geometry of geodesics in Lobachevsky space is discussed in Buseman & Kelly [4], where there is also a proof of Theorem 1. The horospheres in Lobachevsky space are discussed in Gelfand, Graev, and Vilenkin [6]. Lobachevsky's original work can be found in the appendix to Bonola's book [2], where $\pi(\delta)$ is discussed in detail. A short and rigorous course on Lobachevsky geometry is given in Canon *et al* [5].

By Lobachevsky (hyperbolic) space L^3 we mean a real three-dimensional, simply connected, and noncompact space of constant negative curvature equipped with a standard hyperbolic metric. We adopt for L^3 the standard Gaussian curvature $K = -1$. Sometimes in the literature, K takes a value $K = -k^{-2}$, $k > 0$. In that case, the hyperbolic distance δ in formulas (1) and (2) must be replaced by δ/k.

2. CALCULATION OF LOBACHEVSKY'S FUNCTION $\pi(\delta)$

2.1. *Separation of Geodesics and Changes in the Length Scale*

As is known (see, e.g., Milnor [8]), the geometry of manifolds can be expressed in terms of light rays propagating within the manifold under the identification of light rays with geodesics. Objects dual to geodesics in hyperbolic space are horospheres. A horosphere is a surface orthogonal to a congruence of parallel geodesics and tangent to the boundary at infinity. The internal geometry on horospheres in Lobachevsky space is Euclidean. These facts can be found in Gelfand, Graev, and Vilenkin [6] and in Buseman & Kelly [4].

In order to carry out measurements in Lobachevsky space L^3, we have to introduce coordinates. We choose horospherical coordinates as the most useful for our purpose. First, we take an arbitrary point $o \in L^3$ and call it the *origin* or the *reference point*. Next, we choose an arbitrary equivalence class $[\gamma]$ of parallel geodesics γ and select a representative $\gamma \in [\gamma]$ which contains the reference point o. We call this geodesic γ the *reference geodesic* γ_o. The horosphere Ω containing o and orthogonal to $[\gamma]$ is called the *reference horosphere* Ω_o. The orthogonal grid of geodesics $\gamma \in [\gamma]$ and horospheres Ω forms a system of horospherical coordinates in L^3. The horospherical coordinate frame at a common point $p = \gamma \cap \Omega$ of a geodesic γ and a horosphere Ω is given by three mutually orthogonal vectors, one of which is tangent to the geodesic γ and the two others are tangent to the horosphere Ω. In particular, a fixed frame at the origin o is called the *reference frame*. The orientation on any geodesic $\gamma \in [\gamma]$ is chosen in such a way that any point $p \in \gamma$ follows the point p_∞ at the boundary at infinity which defines the class $[\gamma]$.

The duality between geodesics and horospheres in Lobachevsky space is given by Theorem 1.

Theorem 1 (for the proof, see [4]). *If two parallel geodesics γ_o and γ cut segments l_o and l on two parallel horospheres Ω_o and Ω (separated by a distance δ), then the ratio l/l_o is given by:*

$$\frac{l}{l_o} = e^{\delta}. \tag{1}$$

The length ratio l/l_o in (1) can be represented in a different way, which is convenient for further discussion, namely, $l/l_o = (l - l_o + l_o)/l_o = 1 + z$, where $z > 0$ measures the deviation of parallel geodesics due to the distance δ.

In this notation, equation (1) becomes[1]:

$$\delta = \log(1 + z) \tag{2}$$

2.2. *Angle of Parallelism from Experimental Data*

The function $\pi(\delta)$ (the angle of parallelism) was introduced by Lobachevsky by formula (3) below in his work "Geometric Research on the Theory of Parallels" (see the English translation in the appendix to Bonola [2]),

$$\tan\frac{\pi(\delta)}{2} = \exp(-\delta). \tag{3}$$

Equation (3) is not useful for experimental determination of $\pi(\delta)$ since the way of measuring hyperbolic distances in Lobachevsky space is not known. In order to calculate $\pi(\delta)$, we note that, if we take the distance δ in (3) to be equal to the separation between the parallel horospheres Ω_o and Ω in (1), then, combining (1) and (3), we obtain:

$$\tan\frac{\pi(\delta(l/lo))}{2} = \left(\frac{l}{l_o}\right)^{-1}. \tag{4}$$

Equation (3) expresses the angle of parallelism $\pi(\delta)$ versus the length ratio of two segments l_o and l cut by two parallel geodesics γ_o and γ on parallel horospheres Ω_o and Ω separated by the distance δ. In other words, we replaced physically nonmeasurable distance δ by the corresponding ratio l/l_o which is physically measurable. This is the crucial step connecting abstract non-Euclidean geometry to experimental physics.

Next, we must establish a *physical standard of length* to assign the *reference* length l_o to the standard. We must also show an *operational* way to measure the ratio l/l_o. At this point, recall that the General Conference on Weights and Measures in 1983 adopted as a *primary* length standard, or reference length, the wavelength λ of the iodine stabilized HeNe laser, $\lambda_{He-Ne} = 632.99139822$ nm. Thus, in physical metrology, a common practice is to use an appropriate wavelength λ of electromagnetic radiation as the reference length l. We also follow the same metrology based on a wavelength standard.

[1]Equation (2) explicitly gives the relative increase in the length scale $1+z$ versus the hyperbolic distance δ, due to the exponential divergence of geodesics. In the Poincaré ball model of Lobachevskian space, the hyperbolic distance δ is related to the Euclidean distance d by the rule $d = \tanh\delta$. In terms of Euclidean distance, equation (2) becomes $d = \tanh\log(1 + z)$, which was derived in another way in v. Brzeski & v. Brzeski [3].

In hyperbolic space, the only *practical* way to gain information at distances which can be arbitrary large is to use electromagnetic radiation. We locate the source at the reference frame o and choose the source's own *wavelength* λ_o as the *reference length* l_o. Then we consider two parallel geodesics $\gamma_o, \gamma \in [\gamma]$ separated at the reference frame by the distance $l_o = \lambda_o$.

It follows from Theorem 1 (equation (1)) that two parallel geodesics γ_o and γ separated by λ_o along the horosphere Ω_o (at the source frame) are separated by $\lambda > \lambda_o$ at the horosphere Ω (at the observer frame). The spectral shift $(\lambda - \lambda_o)/\lambda_o = z > 0$ is commonly referred as the redshift, and $\lambda = \lambda_o(1 + z)$. The spectral shift z is routinely measured with great accuracy by spectroscopic techniques.

In view of (2), the left-hand side of (5) is actually a function of z, which we denote by $\beta(z)$.

Therefore, we may write

$$\tan\frac{\beta(z)}{2} = \left(\frac{\lambda}{\lambda_o}\right)^{-1}, \tag{5}$$

$$\beta(z) = 2\arctan\left(\frac{\lambda}{\lambda_o}\right)^{-1} = 2\arctan\frac{1}{1+z}, \tag{6}$$

$$\beta(z) = 2\arctan\frac{1}{1+z}. \tag{7}$$

Equation (7) expresses Lobachevsky's angle of parallelism $\beta(z)$ in terms of physical experimental data, namely, the dilation in wavelength z. Geometrical relations leading to (7) are clear from Fig. 1.

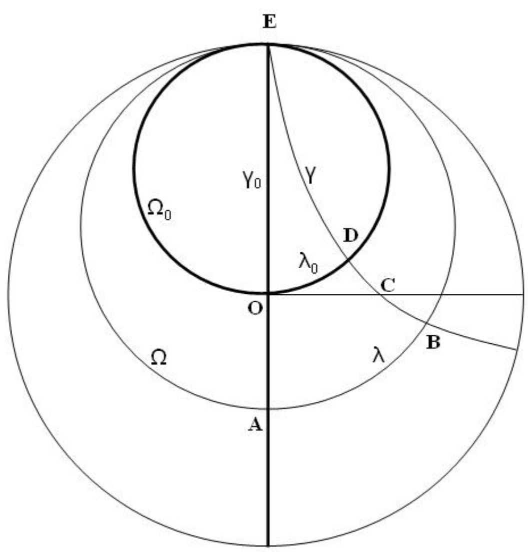

Fig. 1. The Poincaré model of Lobachevsky space in the unit ball model. The geodesics γ_o and γ are parallel. The horospheres Ω_o and Ω are parallel. OA=DB=OC=δ, and OC is the distance which sets the angle of parallelism. The reference geodesic and the reference horosphere are shown in a thicker gauge.

As was mentioned in the introduction, the angle of parallelism is a measure of negative curvature. In particular, if $\beta(z) = 90°$, then the space is Euclidean.[2] One can readily see from (7) that $\beta(z) = 90°$ implies $z = 0$. This simply means that parallel geodesics do not diverge in Euclidean space.

However, astronomical observations of distant objects in space consistently show a spectral shift toward the red part of the spectrum, $z > 0$. The value $z > 0$ (see (7)) implies in turn that $\beta(z) < 90°$, which tells us that space is negatively curved, $K < 0$. Thus, we present strongly supported reasons to believe that the three-dimensional space around us is Lobachevsky space.

[2]Recall that, for an arbitrary k, the right-hand side of (3) is $\exp(-\delta/k)$. We have $K \to 0$ and $\tan(\pi(\delta)/2) \to 1$ as $k \to \infty$.

3. SOME REMARKS

Since the early nineteenth century, when the original formula (3) for $\pi(\delta)$ was discovered by Lobachevsky, it remained an abstract mathematical entity. Its physical realization is $\beta(z)$. We believe that the possibility of evaluating $\beta(z)$ by direct experimental data can attract attention to Lobachevsky geometry, which seems (at least in some cases) to model physics closely. However, already Riemann believed that the actual geometry of physical space can be determined by experiment only; see Riemann [8].

As is well known, three-dimensional Lobachevsky geometry can be realized as the geometry of the coset space $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) = L^3$, where the points of Lobachevsky space L^3 are represented by matrices $a \in \mathrm{SL}(2,\mathbb{C})$, see Gelfand, Graev, and Vilenkin [6]. All equations of Lobachevsky geometry can be obtained in that way. Thus, expressing geometrical entities of Lobachevsky geometry via experimental data, we should also be able to interpret (and name) some group-theoretical entities and constructions by direct experimental data. This is an interesting possibility.

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