An hypersphere model of the Universe – The dismissal of dark matter

Article · March 2004

Source: arXiv

CITATIONS
8

READS
25

1 author:

Jose B. Almeida
University of Minho
99 PUBLICATIONS **979** CITATIONS

SEE PROFILE

An hypersphere model of the Universe – The dismissal of dark matter

José B. Almeida

Universidade do Minho, Departamento de Física, 4710-057 Braga, Portugal.

E-mail: bda@fisica.uminho.pt

Abstract. One can make the very simple hypothesis that the Universe is the inside of an hypersphere in 4 dimensions, where our 3-dimensional world consists of hypersurfaces at different radii. Based on this assumption it is possible to show that Universe expansion at a rate corresponding to flat comes as a direct geometrical consequence without intervening critical density; any mass density is responsible for opening the Universe and introduces a cosmological constant. Another consequence is the appearance of inertia swirls of expanding matter, which can explain observed velocities around galaxies, again without the intervention of dark matter. When restricted to more everyday situations the model degenerates in what has been called 4-dimensional optics; in the paper this is shown to be equivalent to general relativity in all static isotropic metric situations. In the conclusion some considerations bring the discussion to the realm of 4D wave optics.

PACS numbers: 98.80.Jk, 98.62.Dm

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

In this work I make the simple hypothesis that the Universe can be modelled as the volume of an hypersphere in 4 Euclidean dimensions. Naturally the position vector for any point has one single coordinate, the distance to the center of the hypersphere, but displacements have all 4 coordinates: one distance and 3 angles. It is easy to evaluate the length of any displacement and one can easily conclude that for small displacements, provided the distance to the center is large, the angles can be replaced by distances on a plane tangent to an hyperspherical surface, providing a local Euclidean frame for the study of displacements.

In general, though, the hyperspherical nature of the space has important consequences. It is shown that by assigning the meaning of time to the length of displacements one concludes that the distance between any two points in a 3-dimensional hypersurface space increases at a rate proportional to the distance; this is exactly what one finds in our Universe but is derived as a consequence of geometry and not of any critical mass density. A similar argument applied to rotary motion allows the conclusion that this is a natural form of inertial movement and can be applied to galaxies' dynamics to explain the exceedingly large orbital velocities that are detected. Here too, geometry and not hidden mass is the main cause of movement. Naturally mass densities are important for the detailed analysis of observations but they are responsible only for perturbations of a global phenomenon with geometrical causes; it is shown that any mass density is responsible for opening the Universe as well as for a cosmological constant.

On a small scale the space becomes nearly Euclidean and it must be shown that this space is adequate for the description of classical mechanics, at least as effectively as general relativity does; dynamics in Euclidean 4-space is called *4-dimensional optics (4DO)* because it is governed by an extension of Fermat's principle. The paper demonstrates full equivalence between dynamics in hyperbolic general relativity space and 4DO for the case of static isotropic metrics; the particular case of Schwarzschild's metric is analyzed and an exponential metric offering the same predictions as Schwarzschild's is proposed.

2. 4-dimensional hyperspheric coordinates

As an introduction to 4-dimensional hyperspheric coordinates it is useful to revise the case of spherical coordinates in 3 dimensions. The position vector for any point is always written $s = r\sigma_r$, where σ_r is a unitary vector. If needed we can always express σ_r in terms of the orthonormed frame $\{\sigma_1, \sigma_2, \sigma_3\}$

$$\sigma_r = \sin\theta\cos\phi\,\sigma_1 + \sin\theta\sin\phi\,\sigma_2 + \cos\theta\,\sigma_3. \tag{1}$$

We say that $\{\sigma_1, \sigma_2, \sigma_3\}$ is a fiducial frame because it is orthonormed and its vectors don't rotate in a displacement.

A displacement in spherical coordinates is the vector

$$\mathrm{d}s = \partial_r s\,\mathrm{d}r + \partial_\theta s\,\mathrm{d}\theta + \partial_\phi s\,\mathrm{d}\phi, \tag{2}$$

with ∂_μ representing partial derivative with respect to coordinate μ. Resorting to the fiducial frame we can establish the derivatives of σ_r

$$\partial_r\sigma_r = 0, \quad \partial_\theta\sigma_r = \sigma_\theta, \quad \partial_\phi\sigma_r = \cos\theta\,\sigma_\phi, \tag{3}$$

where $\{\sigma_r, \sigma_\theta, \sigma_\phi\}$ form a new orthonormed frame which is not a fiducial frame because its vectors rotate.

$$\sigma_\theta = \cos\theta\cos\phi\,\sigma_1 + \cos\theta\sin\phi\,\sigma_2 - \sin\theta\sigma_3, \tag{4}$$

$$\sigma_\phi = -\sin\phi\,\sigma_1 + \cos\phi\,\sigma_2. \tag{5}$$

We can express this rotation by a set of partial derivatives

$$\begin{aligned} &\partial_r\sigma_r = 0, \quad \partial_\theta\sigma_r = \sigma_\theta, \quad \partial_\phi\sigma_r = \sin\theta\sigma_\phi, \\ &\partial_r\sigma_\theta = 0, \quad \partial_\theta\sigma_\theta = -\sigma_r, \quad \partial_\phi\sigma_\theta = \cos\theta\sigma_\phi, \\ &\partial_r\sigma_\phi = 0, \quad \partial_\theta\sigma_\phi = 0, \quad \partial_\phi\sigma_\phi = -\sin\theta\,\sigma_r - \cos\theta\,\sigma_\theta. \end{aligned} \tag{6}$$

The displacement vector can now be found by application of the derivatives to Eq. (2)

$$\mathrm{d}s = \sigma_r\mathrm{d}r + r\sigma_\theta\mathrm{d}\theta + r\sin\theta\sigma_\phi\mathrm{d}\phi. \tag{7}$$

A coordinate frame for spherical coordinates can't be $\{\sigma_r, \sigma_\theta, \sigma_\phi\}$, however, because the general definition for a coordinate frame is $g_\mu = \partial_\mu s$, [1]. Using Eq. (7) we can write

$$g_r = \sigma_r, \quad g_\theta = r\sigma_\theta, \quad g_\phi = r\sin\theta\,\sigma_\phi. \tag{8}$$

The displacement vector $\mathrm{d}s$ is now written in the general form

$$\mathrm{d}s = g_j\mathrm{d}x^j; \tag{9}$$

where the index j is replaced by (r, θ, ϕ), x^μ represents the coordinates r, θ, ϕ, respectively, and the summation convention for repeated indices is used. Defining the metric tensor elements $g_{jj} = g_j \cdot g_j$ we can evaluate an interval by

$$(\mathrm{d}s)^2 = \mathrm{d}s\cdot\mathrm{d}s = g_{\mu\nu}\mathrm{d}x^j\mathrm{d}x^j. \tag{10}$$

The spherical coordinates example can now be easily extended to a general situation in 4 dimensions. We will consider 4-dimensional space with hyperspheric symmetry where R is the distance to the origin and α^j, $j = 1, 2, 3$ are angles. The position vector is naturally $s = R\sigma_0$, with σ_0 the unit vector of the radial direction; the displacement vector is obtained by extrapolation of Eq. (7)

$$\mathrm{d}s = \mathrm{d}R\sigma_0 + R\left(\mathrm{d}\alpha^1\sigma_1 + \sin\alpha^1\mathrm{d}\alpha^2\sigma_2 + \sin\alpha^1\sin\alpha^2\mathrm{d}\alpha^3\sigma_3\right). \tag{11}$$

If the displacements are small compared to the hypersphere radius R, we can choose a privileged origin for the angles such that all the angles are small and the sines become unity.

$$\mathrm{d}s = \mathrm{d}R\sigma_0 + R\left(\mathrm{d}\alpha^j\sigma_j\right). \tag{12}$$

We will now define the new coordinates $x^j = R\alpha^j$ so that $\mathrm{d}x^j = \mathrm{d}R\alpha^j + R\mathrm{d}\alpha^j$. Inverting the relation it is $R\mathrm{d}\alpha^j = \mathrm{d}x^j - \mathrm{d}Rx^j/R$. Replacing above

$$\mathrm{d}s = \mathrm{d}R\sigma_0 + \left(\mathrm{d}x^j - \frac{\mathrm{d}R}{R}\,x^j\right)\sigma_j. \tag{13}$$

And the displacement length is evaluated by

$$(\mathrm{d}s)^2 = (\mathrm{d}R)^2 + \sum\left(\mathrm{d}x^j - \frac{\mathrm{d}R}{R}\,x^j\right)^2. \tag{14}$$

There is no reason why the displacement should not be given in time units, as long as we use some length and time standards, $\mathcal{L}$ and $\mathcal{T}$ respectively, and replace $\mathrm{d}s = \mathrm{d}t\mathcal{L}/\mathcal{T}$; as a consequence $\mathcal{L}/\mathcal{T} = c$ is the speed of light in vacuum.

$$(\mathrm{d}t)^2 = \left(\frac{\mathcal{T}}{\mathcal{L}}\right)^2\left[(\mathrm{d}R)^2 + \sum\left(\mathrm{d}x^j - \frac{\mathrm{d}R}{R}\,x^j\right)^2\right]. \tag{15}$$

Dividing both members by $(\mathrm{d}t)^2$

$$1 = \left(\frac{\mathcal{T}}{\mathcal{L}}\right)^2\left[(\dot{R})^2 + \sum\left(\dot{x}^j - \frac{\dot{R}}{R}\,x^j\right)^2\right] \tag{16}$$

We are going to interpret the coordinate R as the time elapsed from the Universe's origin, albeit measured as length, and coordinates x^j as being the usual x, y, z coordinates of 3-dimensional space. We will develop the consequences of this interpretation in the following paragraphs.

3. Free space dynamics

Examining displacements on 3D hypersurface we make $\dot{R} = 0$ in Eq. (16)

$$1 = \left(\frac{\mathcal{T}}{\mathcal{L}}\right)^2\sum(\dot{x}^j)^2 = \left(\frac{\mathcal{T}}{\mathcal{L}}\,c\right)^2. \tag{17}$$

Light travels with velocity c in 3-space and the model can accommodate it by zeroing the displacement in the radial direction; photons follow great circles of constant R.

Proceeding to the analysis of massive particle's dynamics we note that the Euler-Lagrange equations for the geodesics of any Riemanian space can be derived from a constant Lagrangian, made equal to 1/2 for convenience [2]. Using Eq. (12) we can evaluate $\mathrm{d}s^2 = \mathrm{d}s\cdot\mathrm{d}s$ and divide both members by $\mathrm{d}s^2$; the first member is then made equal to twice the Lagrangian

$$1 = 2L = \dot{R}^2 + R^2\sum(\dot{\alpha}^j)^2; \tag{18}$$

the four conjugate momenta are

$$p_j = R^2\dot{\alpha}^j = A^j \tag{19}$$

$$p_0 = \dot{R}. \tag{20}$$

The four A^j are conserved quantities because the Lagrangian is independent from α_j; $A^j\sigma_j$ is a vector whose norm is A. Replacing in Eq. (18)

$$1 = \dot{R}^2 + \left(\frac{A}{R}\right)^2. \tag{21}$$

Upon integration, with appropriate choice for the origin of time, we obtain the solution

$$R = \sqrt{A^2 + t^2}, \tag{22}$$

$$\dot{R} = \frac{t}{\sqrt{A^2 + t^2}}. \tag{23}$$

Returning to linear rather than angular coordinates and considering Eq. (19)

$$\dot{x}^j = R\dot{\alpha}^j + \frac{\dot{R}}{R}\,x^j. \tag{24}$$

Inserting Eqs. (19) and (22)

$$\frac{\dot{x}^j}{x^j} = \frac{A^j}{x^j} + \frac{t}{A^2 + t^2}; \tag{25}$$

the first member defines the Hubble parameter H and the second member tells us that the velocity does not stay constant but approximately with tx^j/A^2.

Analyzing Eq. (16) we have to decide if and when the term $\dot{R}x^j/R$ can be neglected in face of $\dot{x}^j$. The condition we want can be expressed by

$$\frac{x^j}{\dot{x}^j} \ll \frac{R}{\dot{R}}; \tag{26}$$

we have a comparison between two times: on the first member the time it would take a distant body to travel to the origin of the laboratory coordinates and on the second member another time which we will assign below to the tame it takes light to travel from the confines of the Universe. This condition is met for nearby objects which are not moving exceedingly slow; when it can be met Eq. (16) reduces to

$$(\dot{R})^2 + \sum(\dot{x}^j)^2 = c^2, \tag{27}$$

placing an upper limit on the speed of moving particles. It is also apparent that the movement of masses implies that they move outwards in the hypersphere through $\dot{R}$.

Returning to Eq. (18) it is easy to conclude that for bodies comoving with the Universe's expansion we must have constant α^j and $\dot{R} = c$, so the Universe must be expanding at the speed of light. For the distance coordinates we get

$$\dot{\alpha}^j = \frac{\dot{x}^j R - \dot{R}x^j}{R^2} = 0. \tag{28}$$

According to the above argument

$$\frac{\dot{x}^j}{x^j} = \frac{\dot{R}}{R} = \frac{c}{R} = H; \tag{29}$$

H is the Hubble parameter and its measurement gives us the size of present day Universe. If we use for the Hubble parameter a value of 81 km s^{-1}/ Mpc the resulting size for the Universe is 1.2×10^{10} ly. Additionally, considering the Universe's expansion is influenced

by its mass density so that A is effectively positive, Eq. (25) tells us the effective Hubble parameter will also increase with time, approximately with t/A^2.

The constant orbital velocity observed in the periphery of most galaxies (ωr = constant) is one of the big puzzles in the Universe which is normally explained with recourse to massive halos of dark matter [3, 4], although some have tried different explanations with limited success; for instance Milgrom [5, 6] modified Newton dynamics empirically. Below we look at the predictions of the hypersphere model for orbital velocities to verify that such explanations are not needed if one accepts that the universe is expanding as an hypersphere.

The gravitational field on the periphery of a galaxy must be negligible without the dark matter halo contribution. The question we will try to answer is whether the Universe expansion can drive a rotation, once the material has been set in motion by some other means. In the affirmative case we must find out if the rotation speed can be kept invariant with distance to the center, as observed in galaxies. Recalling Eq. (13) we will rewrite this equation in spherical coordinates

$$ds = \mathrm{d}R\sigma_0 + \mathrm{d}r\sigma_r + r\mathrm{d}\theta\sigma_\theta + r\sin\theta\mathrm{d}\phi\sigma\phi - \frac{dR}{R}\,r\sigma_r. \tag{30}$$

Notice the last term and compare it to Eq. (13); we have replaced $x^j\sigma_j$ by $r\sigma_r$ in a standard passage from Cartesian to spherical coordinates. It is usual to make $\theta = \pi/2$ whenever dealing with orbits, because we know in advance that orbits are flat. Defining $\mathrm{d}t^2 = \mathrm{d}s^2$ and calling v to $\mathrm{d}s/\mathrm{d}t$ we can write

$$v = \dot{R}\sigma_0 + \left(\dot{r} - \frac{\dot{R}r}{R}\right)\sigma_r + r\dot{\phi}\sigma_\phi. \tag{31}$$

If the parenthesis vanishes the movement becomes circular without any central potential; it is driven solely by the galaxy expanding at the same rate as the Universe. The equation above shows that $r\dot{\phi}$ = constant is the natural inertia condition for the hyperspheric Universe; swirls will be maintained by a radial expansion rate which exactly matches the quotient $\dot{R}/R$. In any practical situation $\dot{R}$ will be very near the speed of light and the quotient will be virtually equal to the hubble parameter; thus the expansion rate for sustained rotation is $\dot{r}/r = H$. If applied to our neighbor galaxy Andromeda, with a radial extent of 30 kpc, using the Hubble parameter value of 81 km s^{-1}/Mpc, as above, the expansion velocity is about 2.43 km s^{-1}; this is to be compared with the orbital velocity of near 300 km s^{-1}.

The model proposed for galaxy dynamics consists of a core dominated by gravitational and electromagnetic interactions from which some material escapes and starts swirling by inertia, while continuing to be accelerated by the remnants of gravity; near the periphery all the gravity is extinct and only inertial rotations prevails.

4. Curved space dynamics

The hypersphere model would be useless if it could not be made compatible with classical mechanics in everyday situations; in this paragraph we will see that full compatibility

exists.

Equation (15) with the constraint $x^j \ll R$ defines 4D Euclidean space, with signature $(++++)$, which differs from Minkowski spacetime with signature $(+---)$. If we use x^0 to represent R the interval of that space is given by

$$(\mathrm{d}t)^2 = \frac{1}{c^2}\sum_{\mu}(\mathrm{d}x^{\mu})^2. \tag{32}$$

In this space Eq. (27) establishes that everything moves with the speed of light and it becomes natural to extend to 4-space Fermat's principle which governs geometric optics in 3D

$$\delta\int_{P_1}^{P_2} n\mathrm{d}s = 0, \tag{33}$$

where n is a function of coordinates 1 to 3, called *refractive index*, defined as the ratio between local 4-speed and the speed of light in vacuum.

$$n = \frac{1}{v} = \frac{\mathrm{d}t}{\mathrm{d}s}. \tag{34}$$

The extension of Fermat's principle to 4D justifies our use of the designation *4-dimensional optics* to refer the study of 4D dynamics and wave propagation; we will use the acronym *4DO* as a substitute for the full designation. From this point onwards we will make $c = 1$ following the uses of general relativity papers, which corresponds to using actual displacements measured in length rather than time units.

In an homogeneous medium Eq. (33) states that trajectories are straight lines in 4-space; in particular when $n = 1$, everything moves with 4-velocity with modulus equal to the speed of light in vacuum. Geometric optics in 3D becomes a direct consequence of 4DO and is obtained from Eq. (33) by setting $\mathrm{d}x^0 = 0$, in agreement with our previous contention that photons travel on 3D space;

$$\delta\int_{R_1}^{R_2} n\mathrm{d}l = 0, \tag{35}$$

with $(\mathrm{d}l)^2 = \sum(\mathrm{d}x^j)^2$ and $j = 1\ldots3$.

The integrand in Fermat's principle, $n\mathrm{d}s$, can be replaced by $\mathrm{d}t$, allowing its interpretation with the phrase: *Radiation and massive bodies travel between two points in 4-space along a path which makes time interval an extremum.* Using $(\mathrm{d}s)^2$ from Eq. (32) the time interval is given by

$$(\mathrm{d}t)^2 = n^2\sum(\mathrm{d}x^{\mu})^2. \tag{36}$$

This can be generalized further without considering non-isotropic media;

$$(\mathrm{d}t)^2 = (n_0\mathrm{d}x^0)^2 + (n_r)^2\sum(\mathrm{d}x^i)^2. \tag{37}$$

The anisotropy relative to coordinate x^0 is not apparent in 3 dimensions, and the medium can still be classified as isotropic. An alternative interpretation of Eq(37) is in terms of interval of curved isotropic space; it is equivalent saying that particles and radiation

travel slower than in vacuum in a given region of space and saying that in the same region space is curved. Following the standard Lagrangian choice

$$1 = 2L = (n_0\dot{x}^0)^2 + (n_r)^2 \sum (\dot{x}^i)^2. \tag{38}$$

The Lagrangian is independent from x^0, so we have a conservation equation

$$(n_0)^2\dot{x}^0 = \frac{1}{\gamma}. \tag{39}$$

Replacing above,

$$1 = \frac{1}{(n_0)^2\gamma^2} + (n_r)^2 \sum (\dot{x}^i)^2. \tag{40}$$

The remaining 3 Euler-Lagrange equations for the trajectory can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \partial_i L; \tag{41}$$

replacing,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(n_r)^2\dot{x}^i\right] = n_0\partial_i n_0(\dot{x}^0)^2 + n_r\partial_i n_r \sum (\dot{x}^j)^2. \tag{42}$$

Expanding the 1st member, inserting Eq. (39) and rearranging

$$\ddot{x}^i = \frac{n_0\partial_i n_0}{n_r^2}(\dot{x}^0)^2 - \frac{\partial_i n_r}{n_r}\sum (\dot{x}^j)^2. \tag{43}$$

The previous equation must now be compared to the predictions of general relativity. A general static relativistic interval for isotropic coordinates can be written

$$(\mathrm{d}s)^2 = \left(\frac{1}{n_0}\right)^2 (\mathrm{d}t)^2 - \left(\frac{n_r}{n_0}\right)^2 \sum (\mathrm{d}x^i)^2. \tag{44}$$

Since n_0 and n_r are arbitrary functions of coordinates x^j, this form allows all possibilities. A suitable Lagrangian for this space's geodesics is

$$2L = 1 = \left(\frac{1}{n_0}\right)^2 \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \left(\frac{n_r}{n_0}\right)^2 \sum \left(\frac{\mathrm{d}x^i}{\mathrm{d}s}\right)^2. \tag{45}$$

There is a conserved quantity because the Lagrangian does not depend on t

$$\frac{1}{(n_0)^2}\frac{\mathrm{d}t}{\mathrm{d}s} = \gamma. \tag{46}$$

Replacing in the lagrangian $\mathrm{d}/\mathrm{d}s \to \mathrm{d}/\mathrm{d}t \times \mathrm{d}t/\mathrm{d}s$ we obtain again Eq. (40):

$$\frac{1}{(n_0)^2\gamma^2} = 1 - (n_r)^2 \sum (\dot{x}^i)^2. \tag{47}$$

We conclude that at least for static isotropic metrics the geodesics of general relativity can be mapped to those of 4DO and so it is a matter of personal preference which formalism each one uses. We believe that the proof can be extended to all static metrics but that is immaterial for the present work.

We will now look at Schwarzschild's metric to see how it can be transposed to 4D optics. We will have to use the dimensionless variable $Gm/(c^2r)$, where G is the gravitational constant. Since a dimensionless variable can be built with $\mathcal{L}m/(\mathcal{M}r)$,

where $\mathcal{M}$ is the mass standard, we will choose $\mathcal{M} = G\mathcal{L}/c^2 = G\mathcal{T}^2/\mathcal{L}$ and avoid constants in the expressions.

The usual form of Schwarzschild's metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{\rho}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{\rho}\right)^{-1}\mathrm{d}\rho^2 - \rho^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{48}$$

This form is non-isotropic but a change of coordinates can be made that returns an isotropic form D'Inverno [7, section 14.7]:

$$r = \left(\rho - m + \sqrt{\rho^2 - 2m\rho}\right)/2; \tag{49}$$

and the new form of the metric is

$$\mathrm{d}s^2 = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2\mathrm{d}t^2 - \left(1 + \frac{m}{2r}\right)^4\left[\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]. \tag{50}$$

This corresponds to the refractive indices

$$n_0 = \frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}, \qquad n_r = \frac{(1 + \frac{m}{2r})^3}{1 - \frac{m}{2r}}, \tag{51}$$

which can then be used by 4DO in Euclidean space.

We turn now to the constraints on the refractive indices so that experimental data on light bending and perihelium advance in closed orbits can be predicted. Light rays are characterized by $\mathrm{d}x^0 = 0$ in 4DO or by $\mathrm{d}s = 0$ in general relativity; the effective refractive index for light is then

$$\sqrt{\frac{1}{\sum(\dot{x}^i)^2}} = n_r. \tag{52}$$

For compatibility with experimental observations n_r must be expanded in series as (see [8])

$$n_r = 1 + \frac{2m}{r} + O(1/r)^2. \tag{53}$$

This is the bending predicted by Schwarzschild's metric and has been confirmed by observations.

For the analysis of orbits its best to rewrite Eq. (38) for spherical coordinates; since we know that orbits are flat we can make $\theta = \pi/2$

$$n_0^2\dot{\tau}^2 + n_r^2(\dot{r}^2 + r^2\dot{\phi}^2) = 1. \tag{54}$$

The metric depends only on r and we get two conservation equations

$$n_0^2\dot{\tau} = \frac{1}{\gamma}, \qquad n_r^2 r^2\dot{\phi} = J. \tag{55}$$

Replacing

$$\frac{1}{\gamma^2 n_0^2} + n_r^2\dot{r}^2 + \frac{J^2}{n_r^2 r^2} = 1. \tag{56}$$

The solution of this equation calls for a change of variable $r = 1/u$; as a result it is also $\dot{r} = \dot{\phi}\mathrm{d}r/\mathrm{d}\phi$; replacing in the equation and rearranging

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{n_r^2}{J^2} - \frac{n_r^2}{J^2\gamma^2 n_0^2} - u^2. \tag{57}$$

To account for light bending we know that $n_r \approx 1 + 2mu$. For n_0 we need 2nd order approximation [8], so we make $n_0 \approx 1 + \alpha mu + \beta m^2 u^2$. We can also assume that velocities are low, so $\gamma \approx 1$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 \approx \frac{2\alpha m}{J^2}\,u + \left(-1 + \frac{8\alpha m^2}{J^2} - \frac{3\alpha^2 m^2}{J^2} + \frac{2\beta m^2}{J^2}\right) u^2. \tag{58}$$

For compatibility with Kepler's 1st order predictions $\alpha = 1$; then, for compatibility with observed planet orbits, $\beta = 1/2$. Together with the constraint for n_0, these are the conditions that must be verified by the refractive indices to be in agreement with experimental data.

We know, of course, that the refractive indices corresponding to Schwarzschild's metric verify the constraints above, however that is not the only possibility. Schwarzschild's metric is a consequence of Einstein's equations when one postulates that vacuum is empty of mass and energy, but the same does not necessarily apply in 4DO. Leaving an open question about what equations should be the counterparts of Einstein's in 4DO, one interesting possibility for the refractive indices, in full agreement with observations, is provided by

$$n_0 = \mathrm{e}^{m/r} \approx 1 + \frac{m}{r} + \frac{m^2}{2r^2}\,, \tag{59}$$

$$n_r = \mathrm{e}^{2m/r} \approx 1 + \frac{2m}{r}\,. \tag{60}$$

These refractive indices are as effective as those derived from Schwarzschild's metric for light bending and perihelium advance prediction, although they do not predict black holes. There is a singularity for $r = 0$ which is not a physical difficulty since before that stage quantum phenomena have to be considered and the metric ceases to be applicable; in other words, we must change from geometric to wave optics approach.

We shall look now at how the overall mass density in the Universe affects its expansion rate; for this we will adopt the refractive indices from Eqs: (59) and (60) and we will denote $n_0 = n$ and $n_r = n^2$. The radial equation Eq. (30) can be used to construct a geodesic equation modified with the introduction of n; we are only interested in radial trajectories so $\mathrm{d}\theta = \mathrm{d}\phi = 0$;

$$c^2 = n^2\dot{R}^2 + n^4\dot{r}^2 + n^4\left(\frac{\dot{R}}{R}\right)^2 r^2. \tag{61}$$

Rearranging and noting that $H = \dot{r}/r$ is the measured Hubble parameter

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{c^2}{n^4} - \frac{\dot{R}^2}{n^2}\right)\frac{1}{r^2} \tag{62}$$

Looking at the definition for n, Eq. (59), we note that m is the mass internal to a sphere of radius r and must remain constant $m = 4\pi\rho_0 r_0^3$, where ρ_0 is the present density and r_0 is the present radius of the sphere. Approximating the exponentials to the first order terms

$$H^2 \approx \left(\frac{\dot{R}}{R}\right)^2 + \frac{c^2 - \dot{R}^2}{r^2} + \frac{(2\dot{R}^2 - 4c^2)m}{r^3}. \tag{63}$$

This equation is similar to Friedman's and can be interpreted as follows: when the mass density is zero the exponentials are unity, we have seen that $\dot{R} = c$ and the Hubble parameter is $H = c/R$; in the original Friedman equation this situation corresponds to a flat Universe and is attributed to a critical density, whose source is attributed to dark matter. Any mass density will make $\dot{R} < c$ and the second term produces an open Universe; the 3rd term is essentially constant because m is proportional to r^3 and corresponds to the cosmological constant.

5. Conclusion

This work is a natural development of speculations I started to make almost 4 years ago about 4DO being an alternative formulation for relativity. At the onset the reasoning was that if one wants to restrict 3-dimensional velocity to the speed of light, a logical thing to do is to postulate a 4th dimension and then state that velocity is always equal to the speed of light but can make different angles to the 4th dimension. If then only the 3-dimensional projection of velocity is considered this can take any value between 0 and the speed of light. I wrote several essays elaborating on that concept which are all available for download from the e-print archive. I made several mistakes along the way but I don't intend to remove the respective essays because they will allow readers to trace the track I've followed. There is one work which I still think is important that people read [9], where a comparison is established between special relativity and 4DO using the method known as K-calculus.

The hypersphere model of the Universe is a generalization of 4DO; it is simpler in terms of basic postulates and incorporates 4DO for everyday situations of classical mechanics. That model is capable of explaining such puzzles as Universe flatness or orbital velocities around galaxies as resulting entirely from geometry, thus avoiding the discomfort of postulating enormous amounts of dark matter. When dealing with classical mechanics problems 4DO was proven to be equivalent to general relativity in all situations characterized by static isotropic metrics and this equivalence is most likely extendable to all static metric situations.

One point that made people react against 4DO in the past was the difficulty in understanding the meaning of coordinate x^0. In fact geodesics of 4DO space can be mapped to those of relativity but the same does not happen with points in both spaces. A point where two relativistic geodesics cross is not mapped to the crossing point of the corresponding geodesics in 4DO. A point in relativistic space is interpreted as an event

and the meaning of points in 4DO space is difficult to grasp. It is important to consider that 4DO is a space for optics so an elementary particle travelling in a given direction with a known momentum should not be interpreted as a trajectory in 4DO but rather as a plane wave that can be represented by any line normal to the wavefronts.

An example taken from optics may clarify the situation. Imagine a plane wave travelling along the x direction and another plane wave travelling at some angle to x. It makes no sense to ask at what position along x the two waves meet because they meet everywhere. However, if these waves were synchronized by some means, for instance if they were split from the same laser beam and then redirected to converge, it would be possible to measure the length travelled by the two waves and there would be a particular position where the two measurements would be equal. In 4DO all trajectories are representative of waves that were essentially all split from the same source when the big bang happened; so even if there is a multitude of lines representing a trajectory it is possible to define events as those points where two measurements along different paths become equal.

In this work we took the approach of trajectories, which is the 4DO equivalent to geometrical optics; in the future it is planned to extend this analysis with the help of wave and Fourier optics in their 4-dimensional extensions.

References

[1] D. Hestenes, *Curvature calculations with spacetime algebra*, Int. J. Theor. Phys. **25**, 581, 1986, URL `http://modelingnts.la.asu.edu/pdf/Curv_cal.pdf`.

[2] J. L. Martin, *General Relativity: A Guide to its Consequences for Gravity and Cosmology* (Ellis Horwood Ltd., U. K., 1988).

[3] J. Silk, *A Short History of the Universe* (Scientific American Library, N. York, 1997).

[4] J. V. Narlikar, *Introduction to Cosmology* (Cambridge University Press, Cambridge, U. K., 2002), 3rd ed.

[5] M. Milgrom, *A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis*, Astrophys. J. **270**, 365, 1983.

[6] M. Milgrom, *A modification of the Newtonian dynamics - implications for galaxies*, Astrophys. J. **270**, 371, 1983.

[7] R. D'Inverno, *Introducing Einstein's Relativity* (Clarendon Press, Oxford, 1996).

[8] C. M. Will, *The confrontation between general relativity and experiment*, Living Rev. Relativity **4**, 4, 2001, URL `http://www.livingreviews.org/Articles/Volume4/2001-4will/`.

[9] J. B. Almeida, *K-calculus in 4-dimensional optics*, 2002, `physics/0201002`.