A complete Leibniz-Mach cosmology: I: The Leibniz Question

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Abstract

The claim that the large scale structure of the Universe is heirarchical has a very long history going back at least to Charlier's papers of the early 20th century. In recent years, the debate has centered largely on the works of Sylos Labini, Joyce, Pietronero and others, who have made the quantative claim that the large scale structure of the Universe is quasi-fractal with fractal dimension $D \approx 2$. There is now a concensus that this is the case on medium scales, with the main debate revolving around what happens on the scales of the largest available modern surveys.

This paper, which is a realization of a worldview which is deeply rooted in the ideas of Leibniz & Mach shows, as a very special case of a general formalism, that such a fractal $D \approx 2$ world, which we denote as $\mathcal{U}_0$ world, is necessarily a world of dynamical equilibrium: that is, it is a *flat* world. Whilst we confine ourselves here to discussing cosmology on the medium scale (that is, $\mathcal{U}_0$ world), the general nature of the derived formalism allows ready specializations to the small-scale world of Newtonian gravitational physics and to the world of very large scale gravitational physics.

To illustrate the utility of the general approach, we derive a simple disc-galaxy model and show how the Tully-Fisher relation and Freeman's Law arise in a very closely related way from a special case of the model.

1 Introduction:

In modern terms, for Leibniz, *space* is a secondary construct projected out of primary relationships between *objects* (whatever these might be) whilst *time* (for Leibniz and Mach) stands as no more than a metaphor for *process* or *ordered change* within material systems. One might reasonably refer to any cosmological synthesis based upon these ideas as a Leibniz-Mach cosmology.

In the debate of Clarke-Leibniz (1715∼1716) (Alexander (1984)) concerning the nature of space, and in which Clarke argued for Newton's conception of an absolute space, Leibniz made three arguments of which the second was:

> *Motion and position are real and detectable only in relation to other objects ... therefore empty space, a void, and so space itself is an unnecessary hypothesis.*

Of Leibniz's three arguments, this latter was the only one to which Clarke had a good objection - essentially that *accelerated motion,* unlike uniform motion, can be perceived without reference to external bodies and is therefore, he argued, necessarily perceived with respect to the absolute space of Newton. Given Clarke's objection, which was pertinent, and whatever Leibniz's actual intention, it is clear that the world implicit to his argument above is a world of dynamical equilibrium (motion everywhere uniform) - otherwise Clarke's objection must stand. So, Leibniz's implied equilibrium state leads us to consider whether or not such a thing can actually exist, and hence we are led to that which we nominate as *The Leibniz Question:*

> *Is it possible to conceive a non-trivial global mass distribution, isotropic about every spatial origin, which is in a state of dynamical equilibrium (motion everywhere uniform) and, if so, what are the properties of this distribution?*

Certainly, Mach himself would have answered *yes* to this question identifying the global mass distribution concerned with the distribution of 'fixed stars', for he famously said

> *... I have remained to the present day the only one who insists upon referring the law of inertia* (Newton's First Law) *to the earth and, in the case of motions of great spatial and temporal extent, to the fixed stars ...* Mach (1919)

where here, he is talking not of that which *accelerates* a particle (a *force*), but rather of that which makes a particle *resistant* to such an acceleration (its *inertia*). Notwithstanding Mach's views on the subject, it is the case that within the world of Newton's Universal Gravitation in which all masses attract all other masses, or within the world of Einstein's spacetime continuum curving in response to a non-trivial mass-energy content, the answer to *The Leibniz Question* question is quite simply *no*.

But we must consider the fact that, in a global context, thermodynamic and dynamic equilibrium are two sides of the same coin and that, in the CBR, we actually see physical evidence of some degree of global thermodynamic equilibrium which should entail a corresponding degree of dynamic equilibrium. So, if the world we observe really is ordered in the manner of the Leibniz-Mach analysis then, given the consistent synthesis of a Leibniz-Mach cosmology, we should expect such a cosmology to yield the answer *yes* in response to the *The Leibniz Question*, together with a description of the associated matter distribution.

The core analysis of this paper, which was motivated by the *The Leibniz Question*, synthesizes such a Leibniz-Mach cosmology and provides the answer *yes* to the question, describing the non-trivial matter distribution concerned as being fractal, $D = 2$. This is entirely consistent with the modern concensus that, on medium cosmological scales at least, material is distributed quasi-fractally, $D \approx 2$, and that on these scales dynamical equilibrium appears to be the dominant state[1]. In recognition of these latter circumstances, we begin with a brief review of the large-scale structure observations and the debate surrounding them.

2 Medium to large scale structure

A basic assumption of the *Standard Model* of modern cosmology is that, on some scale, the universe is homogeneous; however, in early responses to suspicions that the accruing data was more consistent with Charlier's conceptions Charlier (1908, 1922, 1924) of an hierarchical universe than with the requirements of the *Standard Model*, De Vaucouleurs (1970) showed that, within wide limits, the available data satisfied a mass distribution law $M \approx r^{1.3}$, whilst Peebles (1980) found $M \approx r^{1.23}$.

2.1 Modern observations and the debate

The situation, from the point of view of the *Standard Model*, continued to deteriorate with the growth of the data-base to the point that, Baryshev et al (1995) were able to say

> *...the scale of the largest inhomogeneities (discovered to date) is comparable with the extent of the surveys, so that the largest known structures are limited by the boundaries of the survey in which they are detected.*

For example, several redshift surveys of the late 20th century, such as those performed by Huchra et al (1983), Giovanelli and Haynes (1986), De Lapparent et al (1988), Broadhurst et al (1990), Da Costa et al (1994) and Vettolani et al (1993) etc discovered massive structures such as sheets, filaments, superclusters and voids, and showed that large structures are common features of the observable universe; the most significant conclusion drawn from all of these surveys was that the scale of the largest inhomogeneities observed in the samples was comparable with the spatial extent of those surveys themselves.

In the closing years of the century, several quantitative analyses of both pencil-beam and wide-angle surveys of galaxy distributions were performed: three examples are given by Joyce, Montuori & Sylos L (1999) who analysed the CfA2-South catalogue to find fractal behaviour with $D = 1.9 \pm 0.1$; Sylos Labini & Montuori (1998) analysed the APM-Stromlo survey to find fractal behaviour with $D = 2.1 \pm 0.1$, whilst Sylos Labini, Montuori & Pietronero (1998) analysed the Perseus-Pisces survey to find fractal behaviour with $D = 2.0 \pm 0.1$. There are many other papers of this nature, and of the same period, in the literature all supporting the view that, out to $30 - 40h^{-1}Mpc$ at least, galaxy distributions appeared to be consistent with the simple stochastic fractal model

[1] density parameter estimates suggest that our Universe is at least very close to being flat, if not actually flat.

with the critical fractal dimension of $D \approx D_{crit} = 2$.

This latter view became widely accepted (for example, see Wu, Lahav & Rees (1999)), and the open question became whether or not there was transition to homogeneity on some sufficiently large scale. For example, Scaramella et al (1998) analyse the ESO Slice Project redshift survey, whilst Martinez et al (1998) analyse the Perseus-Pisces, the APM-Stromlo and the 1.2-Jy IRAS redshift surveys, with both groups claiming to find evidence for a cross-over to homogeneity at large scales.

At around about this time, the argument reduced to a question of statistics (Labini & Gabrielli (2000), Gabrielli & Sylos Labini (2001), Pietronero & Sylos Labini (2000)): basically, the proponents of the fractal view began to argue that the statistical tools (that is, two-point correlation function methods) widely used to analyse galaxy distributions by the proponents of the opposite view are deeply rooted in classical ideas of statistics and implicitly assume that the distributions from which samples are drawn are homogeneous in the first place. Hogg et al (2005), having accepted these arguments, applied the techniques argued for by the pro-fractal community (which use the *conditional density* as an appropriate statistic) to a sample drawn from Release Four of the Sloan Digital Sky Survey. They claimed that the application of these methods does show a turnover to homogeneity at the largest scales thereby closing, as they see it, the argument. In response, Sylos Labini, Vasilyev & Baryshev (2006) criticized their paper on the basis that the strength of the conclusions drawn is unwarrented given the deficencies of the sample - in effect, that it is not big enough. More recently, Tekhanovich & Baryshev (2016) have addressed the deficencies of the Hogg et al analysis by analysing the 2MRS catalogue, which provides redshifts of over 43,000 objects out to about 300Mpc, using conditional density methods; their analysis shows that the distribution of objects in the 2MRS catalogue is consistent with the simple stochastic fractal model with the critical fractal dimension of $D \approx D_{crit} = 2$.

To summarize, the proponents of non-trivially fractal large-scale structure have won the argument out to medium distances and the controversy now revolves around the largest scales encompassed by the SDSS.

2.2 $\mathcal{U}_0$ world: an equilibrium world on the medium cosmological scale

As the discussion of §2.1 makes clear, it is now generally accepted that, on what might be referred to as the medium scale, the amount of mass within an arbitrarily drawn sphere of radius R varies (in a statistical sense) fractally, $D \approx 2$. That is, defining ρ_0 as a surface mass density parameter then, about any centre, mass is distributed (in a statistical sense) according to

$$\mathcal{M}(R) = 4\pi\rho_0 R^2. \tag{1}$$

At face value, it seems counter-intuitive that matter could or should be distributed in such a way. However, there is a sense in which any distribution of material which is in thermodynamic equilibrium *must* be so distributed: specifically, since (1) is valid about any centre then, on the surface of any sphere drawn in the fractal distribution, the mass surface density is the fixed constant, $\rho_0 > 0$. But the mass surface density on any spherical surface enclosing a volume is directly related to the radiation pressure arising from the radiative activity of any enclosed material. Thus, if one imagines two distinct spheres in $\mathcal{U}_0$ world just touching at a single point, then the net radiation pressure acting across the two surfaces at the point is zero.

In other words, a system organized according to (1) is in radiative/thermodynamic equilibrium. In this way, it is easy to see how, given the freedom, a distribution of material will self-organize according to (1). The conclusions drawn from this simple thermodynamic argument are entirely consistent with (and complementary to) a primary result of this paper - that a world of *dynamical equilibrium* is necessarily associated with a fractal $D = 2$ distribution of material.

3 A brief history of ideas of space and time

The conception of space as the container of material objects is generally considered to have originated with Democritus and, for him, it provided the stage upon which material things play out their existence - *emptiness* exists and is that which is devoid of the attribute of *extendedness* (although, interestingly, this latter conception seems to contain elements of the opposite view upon which we shall comment later). For Newton (1687), an extension of the Democritian conception was basic to his mechanics and, for him:

> ... *absolute space, by its own nature and irrespective of anything external, always remains immovable and similar to itself.*

Thus, the absolute space of Newton was, like that of Democritus, the stage upon which material things play out their existence - it had an *objective existence* for Newton and was primary to the order of things. In a similar way, time - *universal time,* an absolute time which is the same everywhere - was also considered to possess an objective existence, independently of space and independently of all the things contained within space. The fusion of these two conceptions provided Newton with the reference system - *spatial coordinates* defined at a *particular time* - by means of which, as Newton saw it, all motions could be quantified in a way which was completely independent of the objects concerned. It is in this latter sense that the Newtonian conception seems to depart fundamentally from that of Democritus - if *emptiness* exists and is devoid of the attribute of *extendedness* then, in modern terms, the *emptiness* of Democritus can have no *metric* associated with it. But it is precisely Newton's belief in *absolute space & time* (with the implied virtual clocks and rods) that makes the Newtonian conception a direct antecedent of Minkowski spacetime - that is, of an empty space and time within which it is possible to have an internally consistent discussion of the notion of *metric.*

The contrary view is generally considered to have originated with Aristotle (Wicksteed & Cornford (1929) and McKeon (1941)) for whom there was no such thing as a *void* - there was only the *plenum* within which the concept of the *empty place* was meaningless and, in this, Aristotle and Leibniz (Ariew & Garber (1989)) were at one. It fell to Leibniz, however, to take a crucial step beyond the Aristotelian conception: in the debate of Clarke-Leibniz (1715∼1716) (Alexander (1984)) in which Clarke argued for Newton's conception, Leibniz made three arguments of which the second was:

> *Motion and position are real and detectable only in relation to other objects ... therefore empty space, a void, and so space itself is an unnecessary hypothesis.*

That is, Leibniz introduced a *relational* concept into the Aristotelian worldview - what we call *space* is a projection of *relationships* between material bodies (whatever these might be) into the perceived world whilst what we call *time*, which is implied by the idea of *motion*, is the projection of ordered *change* into the perceived world. Of Leibniz's three arguments, this latter was the only one to which Clarke had a good objection - essentially that *accelerated motion,* unlike uniform motion, can be perceived *without* reference to external bodies and is therefore, he argued, necessarily perceived with respect to the *absolute space* of Newton. It is of interest to note, however, that in rebutting this particular argument of Leibniz, Clarke, in the last letter of the correspondence (Alexander (1984)), put his finger directly upon one of the crucial consequences of a relational theory which Leibniz had apparently not realized (but which Mach much later would) stating as absurd that:

> *... the parts of a circulating body (suppose the sun) would lose the* vis centrifuga *arising from their circular motion if all the extrinsic matter around them were annihilated.*

This letter was sent on October 29th 1716 and Leibniz died on November 14th 1716 so that we were never to know what Leibniz's response might have been.

Notwithstanding Leibniz's arguments against the Newtonian conception, nor Berkeley's contemporary criticisms (Ayers (1992)), which were very similar to those of Leibniz and are the direct antecedents of Mach's, the practical success of the Newtonian prescription subdued any serious interest in the matter for the next 150 years or so until Mach himself picked up the torch. In effect, he answered Clarke's response to Leibniz's second argument by suggesting that the *inertia* of bodies is somehow induced within them by the large-scale distribution of material in the universe:

> *... I have remained to the present day the only one who insists upon referring the law of inertia to the earth and, in the case of motions of great spatial and temporal extent, to the fixed stars ...* Mach (1919)

thereby generalizing Leibniz's conception of a relational universe.

4 Concepts of "time" in the Leibniz-Mach perspective

Mach (1919) was equally clear in expressing his views about the nature of time which are, in effect, very similar to those expressed by Leibniz. They each viewed *time* (specifically Newton's *absolute time*) as a meaningless abstraction. All that we can ever do, Mach argued, is to measure *change* within one system against *change* in a second system which has been defined as the standard (eg it takes half of one complete rotation of the earth about its own axis to walk thirty miles). So, on this Machian view, the 'clock' used to quantify the passage of time for a physical system A is simply an independent physical system B which has been arbitrarily defined as the standard clock.

There are two implications arising from this point of view:

- It implicitly identifies one of the requirements of practical time-keeping, that it must be shareable between different observers. So, we have sun dials and we have atomic clocks;
- It similarly implies that, for such shared time-keeping to be possible, system A must, in principle, be equally able to act as the standard clock for system B as the other way around. It is, of course, desirable that we can say of a standard clock that it ticks at a regular rate; but if we possessed only two (different) clocks, then the statement that clock A is more regular than clock B is merely a matter of convention, rather than an absolute fact.

It follows, from the second of these points, that *all* physical systems must keep their own internal time which is uncalibrated against any external clock. Thus, for example, each one of us is aware of a subjective (albeit qualitative) passage of time; so, imagine one is confined within a darkened room then, without being at all aware of *how many hours have passed*, we are aware, nevertheless, that time *has* passed and the mechanism is simply our innate awareness of changes in our internal physical state ... we become hungry, we become tired, a tooth ache comes and a tooth ache goes. These processes cannot form the basis of any objective quantitative, regular, shareable time-keeping process but, nonetheless, they indicate that the passage of time is also intrinsic to the internal workings of a system. So, in answer to the question *how long does is take to walk thirty miles?* a person could correctly answer, using their innate sense of time, that *it takes "walking thirty miles" worth of time.*

It follows from the foregoing considerations that any definition of *time* based upon the Machian argument must be based upon the ideas of:

- every physical system having its own subjective and private internal time-keeping, any one of which can be chosen as the *standard clock* against which all the other systems reckon the passage of time;
- in practice, one such system being chosen as the standard clock, the choice being merely one of convention usually involving a natural cyclic process.

5 Metrical three-space as a relational property

Whilst Mach was clear about the origins of inertia (in the fixed stars), he did not hypothesize any mechanism by which this conviction might be realized and it fell to others to make the attempt - a typical (although very much incomplete) list might include the names of Einstein (1952), Sciama (1953), Hoyle & Narlikar (1964) and Sachs (1982, 1986) for approaches based on canonical ideas of spacetime, and the names of Assis (1999) and Ghosh (2000) for approaches based on quasi-Newtonian ideas, or Barbour (1982), Barbour & Bertotti (1982) for approaches which are deeply rooted in the contemporary discourse of the foundations of physics.

It is perhaps one of the great ironies of 20thC science that Einstein, having coined the name *Mach's Principle* for Mach's original suggestion and setting out to find a theory which satisfied the newly named Principle, should end up with a theory which, whilst albeit enormously successful, is arguably more an heir to the ideas of Democritus and Newton than to the ideas of Aristotle and Leibniz. One only has to consider the special case solution of Minkowski spacetime, which is empty but metrical, to appreciate this fact.

In this paper we do not set out to construct a theory which focuses on some fixed interpretation of what *Mach's Principle* might mean in the way that some of the authors referred to above have. Rather, we look further back and take the general position of Leibniz about the relational nature of space to be our self-evident starting point and consider the question of *spatial metric* within this general conceptualization - that is, how is the notion of invariant *spatial* distance to be defined in the Leibniz-Mach particle universe?

To answer this latter question, we begin by considering the universe of our actual experience and show how it is possible to define an invariant measure for the *radius* of a statistically defined astrophysical sphere purely in terms of the amount of mass it contains (to within a calibration exercise); we then show how the arguments deployed can be extended to define an invariant measure for an *arbitrary* spatial displacement within the statistically defined astrophysical sphere. In this way, we arrive at a general formalism (for which spherical volumes are just a special case) within which a metrical three-space is projected as a secondary construct entirely out of the internal relationships within the primary distribution of universal material.

The question of how *time* arises within this formalism is particularly interesting: the simple requirement that *time* should be defined in such a way that the formalism is conservative has the direct consequence that *time* becomes an explicit measure of ordered change within the system, and is therefore a measure of *internal time* very much as anticipated by Mach's arguments and discussed here in §4.

The overall result is a general all-scales Leibniz-Mach cosmology. We demonstrate its application here on the medium scale, where we find that conditions of dynamical equilibrium are irreducibly associated with a fractal $D = 2$ distribution of material, thereby answering *The Leibniz Question* which refers to the world that Leibniz was effectively considering in his debate with Clarke of 1715∼1716.

5.1 The general argument

Following in the tradition of Aristotle, Leibniz, Berkeley and Mach we argue that no consistent cosmology should admit the possibility of an internally consistent discussion of *empty* metrical space & time - unlike, for example, General Relativity which has the empty spacetime of Minkowski as a particular solution. In this way, we are implicitly accepting Leibniz's view of *space* as a secondary construct somehow projected out of the relationships between 'objects', whatever they might be, and the shared view of Leibniz and Mach that *time* is merely a metaphor for *process* or *ordered change* within material systems. In essence, therefore, the following synthesis provides a quantitative Leibniz-Mach cosmology within which:

- all metric spatial structure is projected entirely out of the internal relationships within the primary material systems, and without reference to any external source or assumptions;
- the notion of the 'elapsed time' is no more and no less than an idealized model for ordered change within the primary material system;

- all concepts of spatial displacement and elapsed time dissolve in the absence of the material system.

So, recognizing that the most simple space & time to visualize is one which is everywhere inertial - in fact the world that Leibniz had in mind in his discussions with Clarke - then our worldview is distilled into *The Leibniz Question:*

> *Is it possible to conceive a non-trivial global mass distribution, isotropic about every spatial origin, which is in a state of dynamical equilibrium (motion everywhere uniform) and, if so, what are the properties of this distribution?*

In pursuit of this question, we shall assume an idealized model universe which is intended to capture what we see as the irreducible features of the our actual universe. And so:

- *it consists of an indefinite number of discrete but identical primitive particles which possess an ordering property which allows us to say that particle $\mathcal{P}_0$ is nearer/further than particle $\mathcal{P}_1$. The redshift properties of galaxies in the real universe are an example of such an ordering property;*

- *within it, 'time' is to be understood, in a qualitative way, as a measure of* process *or* ordered change;

- *within it, the distribution of material particles is statistically isotropic about any point on a sufficiently large scale and over sufficiently large periods of 'time'.*

Note that, although the idea of *space* is implicit in this model, it is not assumed to be a fundamental primary construct which exists independently of the material content.

It is useful to discuss, briefly, the notion of spherical volumes defined on large astrophysical scales in the universe of our experience. Whilst use of such spherical volumes allows for relatively simple arguments, the resulting formalism is general and not restricted to these purely spherical volumes. So, whilst we can certainly give various precise operational definitions of spherical volumes on small scales, the process of giving such definitions on large scales is decidedly ambiguous. In effect, we have to suppose that redshift measurements are (statistically) isotropic when taken from an arbitrary point within the universe and that they vary monotonically with distance on the large scales we are concerned with. With these assumptions, spherical volumes can be defined (statistically) in terms of redshift measurements - however, their radial calibration in terms of ordinary units (such as metres) becomes increasingly uncertain (and even unknown) on very large redshift scales.

5.2 Astrophysical spheres and a mass-calibrated metric for radial displacements

With the foregoing ideas in mind, the primary step taken in answer to the *The Leibniz Question* is the recognition that, on large enough scales in the universe of our experience, the amount of matter, m say, in a given redshift-defined spherical volume will be given by a well-defined monotonic function of the sphere's redshift radius, z. It follows immediately that a generalized radius for any redshift defined astrophysical sphere can be defined in terms of the sphere's contained mass according to:

$$R = G(m) \tag{2}$$

where m is the mass concerned and G is a monotonic increasing function of m satisfying only the condition $G(0) = 0$. Thus, whatever the form of G chosen, we have defined an invariant measurement for the (generalized) radius of an astrophysical sphere which vanishes in the absence of matter.

It follows immediately that an invariant measure of an arbitrary *radial displacement* can be written, purely in terms of mass, as

$$\Delta R = G(m + \Delta m) - G(m) \tag{3}$$

so that, whatever the form of G chosen, we have a metric which follows Leibniz in the required sense (that 'space' is a secondary construct projected out of a 'matter' distribution) for any displacement which is *purely radial.*

The question now becomes: *how can we generalize this idea of a mass-defined invariant measure for a radial displacement into a mass-defined invariant measure for an arbitrary displacement?* In pursuit of this question, we find it useful to consider primitive human experience.

5.3 Qualitative assessments of 'distance traversed' in life

It is instructive to reflect briefly upon how we, as primitive human beings, form qualitative assessments of 'distance traversed' in our everyday lives without recourse to formal instruments.

In effect, as we travel through a physical environment, we use our changing perspective of the observed scene in a given 'elapsed time' to provide a qualitative assessment of 'distance traversed' in that 'elapsed time'. So, briefly, when walking across a tree-dotted landscape, the changing *angular* relationships between ourselves and the trees provides the information required to assess both *distance traversed* and *which tree nearer? which tree further?*, measured in units of human-to-tree angular displacements, within that landscape. If we remove the perspective information - by, for example, obliterating the scene with dense fog - then all sense of 'distance traversed' is destroyed.

So, in making our primitive assessments of *distance traversed* and *which tree nearer? which tree further?*, the primary information required is the concomitant change in the angular relationships between ourselves and everything else in that landscape arising from a spatial displacement.

5.4 The mass model, and its generalization from the sphere

Before we can usefully apply the insights of §5.3, we need to define exactly what object is to be our 'tree dotted landscape': to this end, we consider (2), $R = G(m)$, to be absolutely primary and then invert it to give (as a secondary construct) the derived mass model,

$$Mass \equiv m = M(R), \tag{4}$$

for our rudimentary universe. It is this object which we take to represent our 'tree dotted landscape'. Note that:

- by (2), R vanishes in the case of the rudimentary universe being empty;
- R only becomes calibrated when G becomes defined;
- since $G(0) = 0$ does not imply $M(0) = 0$, then we can retain the freedom for

  $$M(0) = M_0 \geq 0 \tag{5}$$

  so that any given point in the space may, or may not, have an elemental mass (a point-source) situated at it.

In explicit recognition that a 3-space continuum is being admitted, we write (4) as

$$m \equiv M(R), \quad R \equiv f(x^1, x^2, x^3)$$

where R is (as yet) uncalibrated, and we assume nothing about the spatial coordinates, (x^1, x^2, x^3).

Note that, although up to this point, $R = constant$ has been interpreted as a spherical surface for ease of discussion, it can, in fact, represent *any* convex astrophysical surface; for example, an ellipsoid to model an elliptical galaxy; or a cylinder, a slice of which might be taken as a primitive model for a spiral galaxy etc. In this broader context, equation (3) (originally interpreted to define the idea of a mass-calibrated *radial* displacement) is generalized so that it defines a mass-calibrated displacement *normal* to the level surface $R = constant$.

5.5 A mass-calibrated metric for arbitrary spatial displacements

We have a way of assigning a mass-calibrated metric for displacements which are purely normal to the level surface $R = constant$ (of which the sphere is a special case) in our model universe. We now need a way of assigning a mass-calibrated metric for arbitrary displacements within that universe. The reflections of §5.3 on how we, as primitive observers in a landscape, manage this without recourse to formal instruments inform our approach to the problem.

Since we are taking the mass-model $m \equiv M(R)$ to represent the observed landscape of §5.3, then the normal gradient vector

$$n_a = \nabla_a M$$

(which does not require any metric stucture for its definition) represents our perspective upon that landscape - it contains direct angular information and basic information about the local distribution of material along the uncalibrated 'line of sight' normal vector.

Within the primitive human landscape, it was the change in perspective arising from the act of an observer-displacement that produced the information required to assess both *distance traversed* and *which tree nearer? which tree further?*. In the case of our simple model, the change in n_a (the perspective) arising from a displacement dx^k can be formally expressed as

$$dn_a = \frac{1}{8\pi\rho_0} \nabla_i \left(\nabla_a M\right) dx^i \,, \tag{6}$$

where:

- ρ_0 is a characteristic mass surface density inserted to ensure that the dimensions of dn_a are the same as those of dx^a;
- the factor 8π is included for book-keeping purposes;
- we assume that the connection required to give this latter expression an unambiguous meaning is the usual Levi-Civita connection - except of course, the metric tensor g_{ab} required to define that connection in our Leibniz-Mach three-space is not yet defined.

Now, since g_{ab} is not yet defined, then the covariant counterpart of dx^a, given by $dx_a = g_{ai}dx^i$, is also not yet defined. However, providing that $\nabla_a \nabla_b M$ is nonsingular, then (6) provides a 1:1 mapping between the contravariant vector dx^a and the covariant vector dn_a so that, in the absence of any other definition, we can *define* dn_a to be the covariant form of dx^a. In this latter case the metric tensor of our Leibniz-Mach three-space automatically becomes

$$g_{ab} \equiv \frac{1}{8\pi\rho_0}\nabla_a \nabla_b M \equiv \frac{1}{8\pi\rho_0}\left(\frac{\partial^2 M}{\partial x^a \partial x^b} - \Gamma^k_{ab}\frac{\partial M}{\partial x^k}\right), \tag{7}$$

where Γ^k_{ab} are the Christoffel symbols, and given by

$$\Gamma^k_{ab} = \frac{1}{2}g^{kj}\left(\frac{\partial g_{bj}}{\partial x^a} + \frac{\partial g_{ja}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^j}\right).$$

In this way, we arrive at a set of non-linear differential equations defining g_{ab} in terms of the unspecified mass function, $M(R)$. The scalar product

$$ds^2 \equiv dn_i dx^i \equiv g_{ij}dx^i dx^j \tag{8}$$

then provides the required invariant measure for the magnitude of an arbitrary infinitesimal displacement, dx^a, in our Leibniz-Mach three-space.

Comment 1

The crucial point of contact here with Leibniz's view of *space* as a secondary construct projected out of the relationships between 'objects' is simply this: the metric structure of the Leibniz-Mach three-space is defined entirely in terms of the internal properties of the primary material system, represented by $M(R)$, without reference to anything external; and if Leibniz-Mach world is empty, that is if $M \equiv 0$, then the metric structure of the Leibniz-Mach three-space becomes wholly undefined.

Comment 2

Consider Mach's statement:

> *... I have remained to the present day the only one who insists upon referring the law of inertia to the earth and, in the case of motions of great spatial and temporal extent, to the fixed stars ...* Mach (1919)

Given that the 'fixed stars' here find their representation in $M(R)$, then there is an obvious element of this latter statement in (7) above. However, any discussion of *The Law of Inertia* (Newton's First Law) requires a concept of *temporal evolution*, which has yet to be incorporated here. Once it has been so incoprorated, then we can imagine that Mach's statement, above, will find some form of quantitative expression in the present discussion.

6 The metric in terms of the mass function for the spherical special case

We have so far made no assumptions about the nature of the coordinate system (x^1, x^2, x^3). So:

- we suppose that each of (x^1, x^2, x^3) is calibrated in the same units as R;
- noting that cosmic density parameter estimates suggest that, on very large scales, the Universe is at the very least very close to being flat, we make the modelling assumption that it is, in fact, flat so that we can assume the usual Pythagorean relationship,

  $$R^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$$

  so that the level surfaces, $R = constant$ in the model universe are assumed to be spherical.

With this understanding, it is shown, in appendix A, how, for an arbitrarily defined model of mass, $M(R)$, (7) can be exactly resolved to give an explicit form for g_{ab} in terms of such a general $M(R)$: defining the notation

$$\mathbf{R} \equiv (x^1, x^2, x^3), \quad \text{and} \quad M' \equiv \frac{dM}{dR},$$

this explicit form of g_{ab} is given as

$$g_{ab} = \frac{1}{8\pi\rho_0}\left(A\delta_{ab} + Bx^i x^j \delta_{ia}\delta_{jb}\right), \tag{9}$$

where

$$A \equiv \frac{2d_0\,(M - M_0)}{R^2},$$

$$B \equiv -\frac{1}{2R^2}\left(\frac{4d_0\,(M - M_0)}{R^2} - \frac{(M - M_0)'\,(M - M_0)'}{(M - M_0)}\right),$$

where $M_0 \equiv M(0)$ and d_0 is a dimensionless constant of integration. As will eventually become apparent, this constant has a fundamental role to play in the larger Leibniz-Mach worldview.

Finally, noting that *mass* in the above definitions for A and B always appears in the form of $M - M_0$ then we express the formalism exclusively in terms of

$$\mathcal{M}(R) \equiv M - M_0 \tag{10}$$

where, by definition, $\mathcal{M}(0) = 0$, and we refer to $\mathcal{M}(R)$ as the *active mass* of the Leibniz-Mach worldview. So, in terms of this active mass, A and B are defined as:

$$A \equiv \frac{2d_0\mathcal{M}}{R^2}, \tag{11}$$

$$B \equiv -\frac{1}{2R^2}\left(\frac{4d_0\mathcal{M}}{R^2} - \frac{\mathcal{M}'\mathcal{M}'}{\mathcal{M}}\right), \quad \mathcal{M}(0) = 0.$$

The condition $\mathcal{M}(0) = 0$ has the effect that every point in the Leibniz-Mach three-space is equivalent irrespective of whether or not a point-source mass is situated at the point concerned. In other words, $\mathcal{M}(R)$ is always independent of any point-source mass, wherever it might be situated.

With the foregoing definitions, (9) becomes

$$ds^2 \equiv \frac{1}{4\pi\rho_0}\left[\frac{d_0\mathcal{M}}{R^2}dx^i dx^j \delta_{ij} - \left(\frac{d_0\mathcal{M}}{R^2} - \frac{\mathcal{M}'\mathcal{M}'}{4\,\mathcal{M}}\right)dR^2\right], \tag{12}$$

where, from (2), we remember that since $R = G(M)$ for an undefined monotonic function G, then R and hence $\mathcal{M}(R)$ are completely uncalibrated.

7 The temporal dimension

So far, the concept of 'time' has only entered the discussion in a qualitative way in §4 - it has not entered in any quantitative way and, until it does, we cannot talk of *velocities* or *accelerations* or of *equations of motion* ... and hence cannot talk of kinematics or dynamics within $\mathcal{U}_0$ world. In this section, we develop a quantitative definition of time & temporal process in $\mathcal{U}_0$ world.

Since, in its most general definition, *time* is a parameter which orders change within a system, then a necessary pre-requisite for its quantitative definition is a notion of change within the model universe. The most simple notion of change which can be defined in this universe is that of changing relative spatial displacements of the objects within it. Since this universe is populated solely by primitive particles which possess only the property of discrete identity (and hence quantification in terms of the amount of material present) then, in effect, all change can be described as "gravitational" change. In existing classical theories, this fact is incorporated by constraining all particle motions to satisfy the Weak Equivalence Principle. However, this option is not available in the present case, since the WEP is a dynamical principle requiring a prior quantitative definition of "time" and such a definition is still unformulated here.

This latter problem is avoided by formulating a modified version of the WEP which notes that the geometric shapes of gravitational trajectories in ordinary physical space are themselves independent of the internal properties of the particles concerned. So we arrive at the constraint:

> ***C1*** *When a massive test-particle moves under the influence of a gravitating source, the shape of that particle's trajectory in ordinary geometric three-space is independent of the intrinsic properties of the particle concerned;*

7.1 Equations of motion

Suppose p and q are two arbitrarily chosen point coordinates on the trajectory of the chosen particle, and suppose that (8) is used to give the scalar invariant

$$I(p,q) = \int_p^q \sqrt{dn_i dx^i} \equiv \int_p^q \sqrt{g_{ij} dx^i dx^j}. \tag{13}$$

Then, $I(p,q)$ gives a scalar record of how the particle has moved between p and q defined with respect to the particle's continually changing relationship with the mass model, $\mathcal{M}(R)$.

Now suppose $I(p,q)$ is minimized with respect to choice of the trajectory connecting p and q, then this minimizing trajectory, and in particular it's shape, is independent of the internal properties of the particle concerned so that constraint ***C1*** is satisfied.

So, defining the Lagrangian density in the usual way, and using (12), we have

$$I(p,q) = \int_p^q \mathcal{L}\, dt \equiv \int_p^q \sqrt{g_{ij}\dot{x}^i \dot{x}^j}\, dt$$

$$= \left(\frac{1}{\sqrt{8\pi\rho_0}}\right) \int_p^q \left(A\, \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + B R^2 \dot{R}^2\right)^{1/2} dt, \tag{14}$$

$$\text{where} \quad A \equiv \frac{2\, d_0 \mathcal{M}}{R^2}, \quad B \equiv -\frac{1}{2R^2}\left(\frac{4\, d_0 \mathcal{M}}{R^2} - \frac{\mathcal{M}'\mathcal{M}'}{\mathcal{M}}\right)$$

and t is some temporal ordering parameter. At this stage, we note that $\mathcal{I}(p,q)$ is homogeneous degree zero in t which means that it is invariant under $t \to f(t)$ for any monotonic function f. So:

- firstly, since the system is invariant under $t \to f(t)$, then t cannot be identified as physical time;
- secondly, since the system $\mathcal{I}(p,q)$ is homogeneous degree zero in t then, by a standard result, the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{R}}\right) - \frac{\partial \mathcal{L}}{\partial R} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{15}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

  the solutions of which minimize $\mathcal{I}(p,q)$, are not linearly independent. It follows that additional information is required to close the system. A similar circumstance arises in General

Relativity when the equations of motion are derived from an action integral which is formally identical to (13). In that case, the system is closed by specifying the arbitrary time parameter to be 'particle proper time', so that

$$d\tau = \mathcal{L}(x^j, dx^j) \quad \rightarrow \quad \mathcal{L}(x^j, \frac{dx^j}{d\tau}) = 1, \tag{16}$$

which is then considered as the necessary extra condition required to close the system. In the present circumstance, an obvious solution presents itself when we have written down the equations of motion explicitly.

For the sake of simplicity in this work, we shall assume that we remain in the equatorial plane, $\phi = \pi/2$, so that the third Euler-Lagrange equation above is redundant. That said, the radial equation clearly gives rise to two distinct classes of motion: a non-degenerate state class of strictly non-circular motions, and a degenerate state class of purely circular motions. We consider these in turn below.

7.2 The non-degenerate state case: non-circular motions

For the non-degenerate state of strictly non-circular motions - that is, the case of circular motions $R = constant$ being explicitly excluded - the Euler-Lagrange equations can be combined to give the equations of motion in standard vector form as:

$$2A\ddot{\mathbf{R}} + \alpha\,\dot{\mathbf{R}} + \beta\,\mathbf{R} = 0, \tag{17}$$

where

$$\alpha \equiv \left(2A'\dot{R} - 2\frac{\dot{\mathcal{L}}}{\mathcal{L}}A\right)$$

$$\beta \equiv \left(B'R\dot{R}^2 + 2B\left(\dot{R}^2 + R\ddot{R}\right) - \frac{A'}{R}\left(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\right) - 2\frac{\dot{\mathcal{L}}}{\mathcal{L}}BR\dot{R}\right).$$

For the reasons stated in §7.1, this system does not form a linearly independent set, and so needs additional information to close it. Since this circumstance is intimately linked with the idea that the temporal parameter in this system cannot be identified with physical time, then the required additional information must amount to *defining* physical time for the system.

The way forward is fairly obvious: if we wish the system (17) to be conservative then the dissipative term must necessarily disappear: that is, the condition

$$\alpha \equiv \left(2A'\dot{R} - 2\frac{\dot{\mathcal{L}}}{\mathcal{L}}A\right) = 0$$

must hold. It follows immediately that

$$\frac{A'}{A}\dot{R} = \frac{\dot{\mathcal{L}}}{\mathcal{L}} \quad \rightarrow \quad \mathcal{L} = \frac{v_0}{8\pi\rho_0}A, \tag{18}$$

where v_0 is an undetermined parameter with units of *velocity* and ρ_0 is the characteristic mass surface density parameter introduced at equation (6), and the factor 8π is introduced for book-keeping purposes. We shall expand upon the detailed consequences of this in §7.3.

7.3 Internal clocks for the non-degenerate state case:

The condition (18) is directly analogous to (16), the condition which defines 'particle proper time' in General Relativity, and in a similar way removes the arbitrariness in the temporal parameter of (17) by defining *physical time* for the system. To make this explicit, then from (18) and (14) we have

$$\mathcal{L}^2 \equiv g_{ij}\dot{x}^i\dot{x}^j = \frac{v_0^2}{64\pi^2\rho_0^2}A^2 \tag{19}$$

so that the elapsed physical time arising from a spatial displacement as measured by the system's internal clock is *defined* by the relation

$$dt^2 \underset{\text{def}}{=} \frac{1}{v_0^2}\left(\frac{64\pi^2\rho_0^2}{A^2}\right) g_{ij}dx^i dx^j \rightarrow$$

$$dt^2 \underset{\text{def}}{=} \frac{1}{v_0^2}\left(\frac{8\pi\rho_0}{A^2}\right)\left[A\left(dR^2 + R^2 d\theta^2\right) + BR^2 dR^2\right], \tag{20}$$

from which we now see that v_0, which has dimensions of *velocity*, is, in fact, a conversion factor between units of *distance* and units of *time*, which is precisely the interpretation that Bondi attached to the light-speed constant, c, in the 1960s. We shall refer to v_0 as the *clock-rate parameter* and, for now, leave its value unassigned.

In any event, according to the above, the elapsing of internal time for the system concerned is given a direct physical interpretation in terms of the process of *spatial displacement* within that system. Thus, just as we have shown that "metrical space" can be considered to be projected out of the relationships within the material world, so "elapsed system time" can be considered to be projected out of "process" within that world, which conforms exactly with the Leibniz-Mach view on the nature of "time" expressed in §4.

With this understanding, (20) can be written as

$$\dot{R}^2 + R^2\dot{\theta}^2 = \frac{v_0^2}{8\pi\rho_0}A - \frac{B}{A}R^2\dot{R}^2. \tag{21}$$

Following the algebra of §C which eliminates the right-hand-side time derivative, we find:

$$\dot{R}^2 + R^2\dot{\theta}^2 = 4\, d_0^2 v_0^2 \left(\frac{\mathcal{M}}{4\pi\rho_0} - \frac{h^2}{d_0 v_0^2}\right)\left(\frac{\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'}\right) + \frac{h^2}{R^2} \tag{22}$$

where h is conserved angular momentum. This latter relationship, being a restatement of (20), is effectively the *definition* of physical time for the system concerned and is therefore most easily considered as a constraint that must be satisfied by any solution of the completed equations of motion, considered below.

Comment

Again, we recollect that so far R and $\mathcal{M}(R)$ are completely uncalibrated, so the above definition of physical time only becomes specific when that calibration is completed.

7.4 The degenerate state case: circular motions

For this case, the radial equation of (15) is trivially integrated to give:

$$\mathcal{L}^2 = v_0^2 \quad \rightarrow \quad R^2\dot{\theta}^2 = \frac{8\pi\rho_0 v_0^2}{A} \equiv \frac{v_0^2}{d_0}\left(\frac{4\pi\rho_0 R^2}{\mathcal{M}}\right) \tag{23}$$

for some constant $v_0 > 0$ having dimensions of velocity. The transverse equation of motion gives angular momentum conservation, trivially. Note that this system is *not* invariant under $t \rightarrow f(t)$, and so the temporal parameter within it must already correspond to physical time. All this really means is that systems in states of pure circular motion are, by definition, already acting as perfect standard clocks.

8 Every particle is an angular momentum preserving clock

In the following, we show that the total independent content of the completed equations of motion (that is, equation (17) with $\alpha = 0$) arising from the non-degenerate state case is:

- equation (20) which defines elapsed physical time in terms of the process of *spatial displacement*,
- a statement of the conservation of angular momentum,

from which it follows that every particle in the ensemble of particles represented by the active mass function, $\mathcal{M}(R)$, is no more and no less than an angular momentum preserving clock. Since all particles in this ensemble are equivalent, we shall refer to each such clock as a *system clock*.

We demonstrate this as follows: referring to (22) as the *Clock constraint*, the completed equations of motion arising from the non-degenerate state case are given by:

$$2A\ddot{\mathbf{R}} + \beta\,\mathbf{R} = 0 \tag{24}$$

where

$$\beta \equiv \left(B'R\dot{R}^2 + 2B\left(\dot{R}^2 + R\ddot{R}\right) - \frac{A'}{R}\left(\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}\right) - 2\frac{A'}{A}BR\dot{R}^2\right)\mathbf{R}$$

Clock constraint

$$\dot{R}^2 + R^2\dot{\theta}^2 = 4\,d_0^2 v_0^2\left(\frac{\mathcal{M}}{4\pi\rho_0} - \frac{h^2}{d_0 v_0^2}\right)\left(\frac{\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'}\right) + \frac{h^2}{R^2},$$

and where A and B are defined at (11). Following appendix §B, in an analysis which makes explicit use of the condition $R \neq constant$, we find that this latter system can be written in potential form as

$$\ddot{\mathbf{R}} = -\frac{d\mathcal{V}}{dR}\hat{\mathbf{R}}, \tag{25}$$

$$\mathcal{V} = -\frac{1}{2}\left[4\,d_0^2 v_0^2 \left(\frac{\mathcal{M}}{4\pi\rho_0} - \frac{h^2}{d_0 v_0^2}\right)\left(\frac{\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'}\right) + \frac{h^2}{R^2}\right]$$

Clock constraint:

$$\frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2\right) \;=\; -\mathcal{V}$$

from which it is clear that the *Clock constraint* is, in fact, simply the first integral with respect to t of (25) with the condition that the constant of integration which arises from this process is explicitly set to zero.

Since angular momentum is trivially conserved by (25), it follows immediately that the total independent content of these equations is, as stated, an equation (the *Clock constraint* which is equivalent to (20)) defining elapsed physical time in terms of *process* or *change* and angular momentum conservation. In this way, the proposition, that every particle in the ensemble is a *system clock* is demonstrated.

Comment

At face value, the condition that the constant of integration associated with $\mathcal{V}$ must be explicitly set to zero appears to limit the range of solutions available, but this is not the case for the following reason: the additional degree of freedom which is apparently lost by this latter requirement can simply be built into the active mass function in the first instance, so that the given form of the potential function, above, remains invariant. In other words, the active mass function must necessarily have the structure $\mathcal{M} \equiv \mathcal{M}(R, h, \omega)$, where (h, ω) are orbital parameters which determine orbits uniquely. In effect, therefore, the active mass function - and hence the potential function also - becomes a function of dynamical state, containing instructions informing how a particle in a motional state (h, ω) must 'see' the active mass distribution.

8.1 A system clock is synchronous to a classical standard clock

In the following we show, as a trivial mathematical result, that a system clock defined by the *Clock constraint* of (25) is entirely equivalent to a classically defined standard clock, with each exactly synchronized to the other.

The Leibniz-Mach point of view

The *Clock constraint* of (25) can be simplified and rearranged to define, in the Leibniz-Mach sense, an elapsed temporal interval in terms of a spatial displacement as:

$$dt \underset{\text{def}}{=} \left[4\,d_0^2 v_0^2 \left(\frac{\mathcal{M}}{4\pi\rho_0} - \frac{h^2}{d_0 v_0^2}\right)\left(\frac{\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'}\right)\right]^{-1/2} dR \qquad (26)$$

which, by definition, is also a rearranged form of (20) which defines elapsed time as measured by the system's internal clock.

The classical point of view

From the classical perspective, we interpret the parameter t occurring in the equation of motion component of (25) as the *clock time* measured by an external classical clock, ticking off standard units of time, *seconds, minutes, hours* etc. The *Clock constraint* component of (25) is correspondingly re-interpreted as the classical *energy equation* arising from the first integration of the equation of motion wrt classical clock-time, and where the arbitrary constant arising from this process is set identically to zero. This *energy equation* can now be rearranged to give the spatial displacement corresponding to a given elapsed classical clock-time interval as:

$$dR = \left[4\, d_0^2 v_0^2 \left(\frac{\mathcal{M}}{4\pi\rho_0} - \frac{h^2}{d_0 v_0^2} \right) \left(\frac{\mathcal{M}^2}{R^4 \mathcal{M}' \mathcal{M}'} \right) \right]^{1/2} dt. \tag{27}$$

Note that, unlike (26), this is not a definition but is rather the application of a classical equation of motion to determine a spatial displacement in terms of a time-interval determined by an external classical clock.

Conclusions

It is trivially true that (26) and (27) are, in formal terms, simple rearrangements of each other so that if a given (dR, dt) satisfies either one of these expressions, then it must necessarily satisfy the other. In other words the Leibniz-Mach system clock is synchronous to any external classical standard clock used to measure the passage of classical time for the system.

Comment

Whilst this result is mathematically trivial, it is a non-trivial result from the point of view of physics for it is an exact formal realization of the Leibniz-Mach assertion that the notion of *time passing* is no more, and no less, than an idealized model for the process of *ordered change* within a material system. From this point of view, classical clock-time (that is, *external time*) is an un-necessary abstraction and, arguably, is an idea which has led gravitational physics in particular into the conceptual *cul-de-sac* of the spacetime continuum.

8.2 The assignment of value to the clock-rate parameter v_0

As soon as we re-interpret (25) as a classical particle equation of motion (as above), then it becomes clear how the value of the clock-rate parameter, v_0, must be assigned: specifically, when the classical interpretation is employed, the Weak Equivalence Principle (WEP) becomes applicable, which means that particle accelerations must be independent of any properties that the particle might possess. It follows that, for any given particle, the value of v_0 (which has units of *velocity*) must be assigned accordingly. There are then two cases for (25) on the classical interpretation:

- $\mathcal{V} \neq$ *constant*, so that particle accelerations are non-trivial. The WEP can then only be satisfied if the parameter v_0 is independent of the particle motion. This implies that v_0 must be a global constant so that $v_0 \equiv c$ is indicated;

- $\mathcal{V} = constant$, so that particle accelerations are identically zero implying, in turn, that the WEP imposes no constraints on the parameter v_0. However, as we shall show in §9.1, for this case the *Clock constraint* becomes

  $$\dot{R}^2 + R^2\dot{\theta}^2 = v_0^2$$

  which, on the classical interpretation of (25), is the particle's energy equation implying that v_0 must be identified with the velocity magnitude of the particle concerned.

However, whilst the classical interpretation of (25) has indicated how the value of v_0 should be assigned in each of the two situations, we must remember that, within the Leibniz-Mach synthesis, v_0 is the clock-rate parameter which acts as a conversion factor between units of distance and units of time, and is not a velocity magnitude at all. Thus:

- in the case of $\mathcal{V} \neq constant$ for which we have $v_0 = c$, then the light 'velocity' parameter c receives Bondi's interpretation of the 1960's;
- in the case of $\mathcal{V} = constant$, it is easily seen that assigning v_0 values according to particle velocity magnitudes on the classical interpretation is equivalent, in the Leibniz-Mach synthesis, to assigning v_0 values according to the condition that all Leibniz-Mach system clocks are synchronized to *tick* at the same rate.

9 Specific example: $\mathcal{U}_0$ world and the Leibniz Question

According to the discussion of §2.1, we can reasonably suppose that, on the medium cosmological scale at least, matter is distributed according to

$$\mathcal{M}(R) \approx 4\pi\rho_0 R^2 \tag{28}$$

valid about any centre, so that the Copernican Principle holds - in other words, as a non-space filling fractal $D \approx 2$ distribution. In §2.2, we gave a qualitative argument to the effect that such a distribution corresponds to a state of thermodynamic equilibrium so that, given the freedom, it can be expected that any material system will self-organize into such a state. Since, in an ideal world, thermodynamic equilibrium entails dynamical equilibrium, this suggests that the answer to *The Leibniz Question* of §5.1 is *yes*, and that the mass distribution concerned is given by (28).

However, in practice, of course, we know that the simple statement that mass is distributed according to (28) (without the verbal qualifying condition 'valid about any centre') contains a hidden degree of freedom which corresponds to the question: what proportion of it is invariant under rotations *and* translations (and therefore represents a fractal $D = 2$ component), and what proportion of it is invariant *only* under rotations (and therefore represents a distribution with a specific center, $R = 0$)?

So, the immediate question is: if we define $\mathcal{M}(R)$ according to (28), how does this degree of freedom manifest itself in the formalism? To answer this question, consider the general line element, given from (12) as

$$ds^2 \equiv \frac{1}{4\pi\rho_0}\left(\frac{d_0\mathcal{M}}{R^2}dx^i dx^j \delta_{ij} - \left(\frac{d_0\mathcal{M}}{R^2} - \frac{\mathcal{M}'\mathcal{M}'}{4\,\mathcal{M}}\right)dR^2\right)$$

which, for the active mass function (28), becomes:

$$ds^2 \equiv d_0 \, dx^i dx^j \delta_{ij} + (1 - d_0) \, dR^2. \tag{29}$$

Since the component $d_0 \, dx^i dx^j \delta_{ij}$ is invariant under translations and rotations, then it is that part of the line element which is associated with the fractal component of (28). Thus, we can conclude that the parameter d_0, which first appeared at (9), performs the role of partitioning (28) according to

$$\mathcal{M}(R) \equiv d_0 \left(4\pi\rho_0 \, R^2\right)_{FRAC} + (1 - d_0) \left(4\pi\rho_0 R^2\right)_{ROT} \tag{30}$$

in an obvious notation. Note that this interpretation of d_0 implies the constraint $0 \leq d_0 \leq 1$.

With this understood, we find that using (28) in (25) gives:

$$\ddot{\mathbf{R}} = -\frac{d\mathcal{V}}{dR} \hat{\mathbf{R}}, \tag{31}$$

$$\mathcal{V} = -\frac{1}{2} \left[d_0^2 v_0^2 + (1 - d_0) \frac{h^2}{R^2} \right]$$

Clock constraint:

$$\frac{1}{2} \left(\dot{R}^2 + R^2 \dot{\theta}^2 \right) = -\mathcal{V}.$$

It is quite clear that there are two basic cases to consider, these being:

- $0 \leq d_0 < 1$ which gives rise to the $\mathcal{V} \neq constant$, $v_0 = c$ case considered in §8.2;
- $d_0 = 1$ which gives rise to the $\mathcal{V} = constant$, $0 \leq v_0 \leq c$ case also considered in §8.2.

We shall consider the $d_0 = 1$ case in moderate detail below and comment briefly on the $d_0 < 1$ case in §10.

Comment

It is, perhaps, worth noting that the allowed values (above) for the parameter pair (v_0, d_0) form a continuum in the (v_0, d_0) plane, with the consequence that the two distinct cases, $\mathcal{V} \neq constant$ and $\mathcal{V} = constant$ merge smoothly through the point $(v_0, d_0) = (c, 1)$ in the parameter space.

9.1 Case $d_0 = 1$: The Leibniz Question answered

The most simple case occurs when $d_0 = 1$ for which (29) becomes

$$ds^2 \equiv g_{ij} dx^i dx^j = dx^i dx^j \delta_{ij}$$

and (30) becomes

$$\mathcal{M}(R) \equiv \left(4\pi\rho_0 \, R^2\right)_{FRAC}$$

so that the distribution is purely fractal. We find that (25) gives:

$$\ddot{\mathbf{R}} = 0$$

$$\mathcal{V} = -\frac{v_0^2}{2}$$

Clock constraint:

$$\dot{R}^2 + R^2\dot{\theta}^2 \;=\; v_0^2$$

where, according to the considerations of §8.2, the value of the clock-rate parameter v_0 is identified with the classical particle velocity magnitude. This has the effect of ensuring that all Leibniz-Mach system clocks (particles on their trajectories in the classical view) *tick* at the same rate. Thus, we can say:

- For the particular cases $0 \leq v_0 < c$ we have finally answered *The Leibniz Question* and shown that a world of dynamical equilibrium can be associated with a non-trivial global matter distribution, and that this distribution is necessarily purely fractal with $D = 2$. Thus, $\mathcal{U}_0$ world in the special case of $d_0 = 1$ can be consistently identified with the medium scale world observed and discussed in §2.1.

- In the special limiting case of $v_0 = c$, the particles concerned must be massless so that, since they form an equilibrium distribution, we can identify them as composing a blackbody radiative background.

Finally, therefore, on the classical interpretation, we have a cosmos in which all particle motions are uniform with velocities satisfying only the condition $0 \leq v_0 < c$, within which matter is distibuted fractally, $D = 2$, and with an associated blackbody radiation background. In summary, *The Leibniz Question* is completely answered by the special case $d_0 = 1$.

9.2 Mach's Principle: What is it? Where is it?

Mach's statement

> *... I have remained to the present day the only one who insists upon referring the law of inertia* (Newton's First Law) *to the earth and, in the case of motions of great spatial and temporal extent, to the fixed stars ...* Mach (1919)

is a succint expression of that idea for which Einstein coined the phrase *Mach's Principle* and which many, over the years, have interpreted to mean that, somehow or other, inertia is induced in any mass by the action of all other masses in the universe. The very vagueness of the idea, as expressed above, is probably why equally many over the years have failed in their attempts to quantify it in any meaningfull sense.

But having posed *The Leibniz Question*:

> *Is it possible to conceive a non-trivial global mass distribution, isotropic about every spatial origin, which is in a state of dynamical equilibrium (motion everywhere uniform) and, if so, what are the properties of this distribution?*

and resolving it in the manner of §9.1 we see that, interpreting this resolution in classical terms, the *Law of Inertia* is indeed to be referred to Mach's *fixed stars* (although not, perhaps, in the sense that he imagined), so long as these are identified with the observed fractal $D \approx 2$ material distribution of the medium scale cosmos. However, for the purpose of answering the question *What is Mach's Principle?* in a more general way, any attempt to interrogate Mach's statement above in any detailed sense leads to a conceptual cul-de-sac for the reason that, implicit to its formulation, are classical ideas of *time, velocity* and *force*, none of which have survived in the Leibniz-Mach synthesis described herein, except as secondary aids to understanding.

So, if we are to provide an answer to the *Mach's Principle* question, we can probably do no better than identifying it with the fundamental principles upon which the present Leibniz-Mach synthesis is based. We are led to:

> *Mach's Principle - a conjunction of two propositions: firstly, that metrical space is a secondary construct projected out of the internal relationships between the objects in an ensemble of objects; secondly, that the notion of time passing is no more and no less than an idealized model for the process of ordered change within the ensemble.*

10 A simple model for disc galaxies

The special $\mathcal{U}_0$-world case of $d_0 = 1$ corresponds to the purely fractal $D = 2$ equilibrium world,

$$\mathcal{M}_F(R) \equiv \left(4\pi\rho_0\, R^2\right)_{FRAC} \tag{32}$$

where ρ_0 is the characteristic mass surface density of this equilibrium world.

Noting that the gravitation constant G has units of ($acceleration/mass\ surface\ density$) then, corresponding to the characteristic mass surface density $\rho_0 > 0$ scale of $\mathcal{U}_0$ world, there must be a corresponding characteristic acceleration scale

$$a_0 = 4\pi\rho_0\, G. \tag{33}$$

But it is precisely the idea of there being a critical boundary between the internal and external dynamical environments of disc galaxies, defined by an acceleration scale, which underpins the MOND algorithm put forward by Milgrom (1983a,b). This algorithm has had many, very difficult to discount, successes across the general area of disc galaxy phenomonology. See Sanders (2014) for a detailed historical account. The fact that $\mathcal{U}_0$ world forms, by definition, the external environment of disc galaxies combined with the fact that this world also has an associated characteristic acceleration scale, a_0 defined at (33), raises the obvious hypothesis that:

- the boundary between the interior environment of disc galaxies and the exterior environment of $\mathcal{U}_0$ world is identical to the MOND critical acceleration boundary;
- the characteristic acceleration scale of $\mathcal{U}_0$, defined at (33), is the MOND critical acceleration parameter;
- the degenerate-state case of §7.4, which allows for circular motions only, provides the basis for a simple model for the mass distributions and internal motions of disc galaxies.

10.1 The underlying theory content

To construct the disc-galaxy model, we begin by assuming the existence of an unspecified spherically symmetric mass perturbation, $\mathcal{M}(R)$ say, of the equilibrium environment, $\mathcal{M}_F(R)$. By definition, for such a system,

$$\mathcal{M}(R) \rightarrow \mathcal{M}_F(R) \equiv 4\pi\rho_0 R^2 \quad \text{as} \quad \mathrm{R} \rightarrow \infty.$$

In the degenerate-state case of §7.4, the dynamics associated with an arbitrary $\mathcal{M}(R)$ admit only circular motions, given by

$$V_{rot}^2(R) = V_\infty^2 \, \frac{4\pi\rho_0 R^2}{\mathcal{M}(R)}. \tag{34}$$

Clearly, $V_{rot} \rightarrow V_\infty$ as $R \rightarrow \infty$, so that V_∞ is an asymptotic flat rotation velocity.

Now, whilst a disc is, by definition, not spherical, it does sit within its external environment which is spherical. So, our very simple model assumes spherical symmetry, and that all motions take place within the equatorial plane. Equation (34) is then the foundation of the disc-galaxy model.

10.2 The MOND phenomenological content

If we now consider the evidence of MOND that there is a critical acceleration parameter, a_0 say, related to the critical surface density parameter by

$$\rho_0 = \frac{a_0}{4\pi G}$$

and a corresponding critical radius, R_0, at which the critical acceleration is reached, then we can hypothesize that R_0 defines the boundary between the interior environment of disc galaxies and the exterior equilibrium environment. On the basis of this hypothesis, we can deduce that $\mathcal{M}(R)$ in the disc-model of (34) must have the general structure:

$$\begin{aligned} \mathcal{M}(R) &= \mathcal{M}_g(R), \;\; R < R_0 \\ \mathcal{M}(R) &= \mathcal{M}_g(R_0) + 4\pi\rho_0 \left(R^2 - R_0^2\right), \;\; R \geq R_0 \end{aligned}$$

where $\mathcal{M}_g(R)$ is the model for the mass distribution within the galaxy interior. Thus, for this general structure, (34) becomes

$$\begin{aligned} V_{rot}(R) &= V_\infty \left(\frac{4\pi\rho_0 R^2}{\mathcal{M}_g(R)} \right)^{1/2}, \qquad R < R_0 \\ V_{rot}(R) &= V_\infty \left(\frac{4\pi\rho_0 R^2}{\mathcal{M}_g(R_0) + 4\pi\rho_0 \left(R^2 - {R_0}^2\right)} \right)^{1/2}, \qquad R \geq R_0 \end{aligned} \tag{35}$$

where $m_0 \equiv \mathcal{M}_g(R_0)$ is the total mass contained within $R \leq R_0$ and is consequently the total mass of the galaxy concerned.

10.3 The Tully-Fisher relation & Freeman's Law

We use the disc-model to derive Freeman's Law and the baryonic Tully-Fisher relation for a very specific class of galaxies defined by an hypothesized property of some rotation curves. Given the significance of the Tully-Fisher relation and Freeman's Law to modern astrophysics then, in effect, the following amounts to the prediction that the hypothetical class of rotation curves actually does exist.

We remember that $R = R_0$ is the critical acceleration boundary, and is therefore also the boundary between the galactic interior and the exterior environment, and that m_0 is the total mass contained within $R \leq R_0$ and is therefore the total galactic mass.

From the general model (35), we have directly that

$$V_0^2 = V_\infty^2 \left(\frac{4\pi \rho_0 R_0^2}{m_0} \right) \equiv V_\infty^2 \left(\frac{a_0 R_0^2}{G m_0} \right) \tag{36}$$

where $m_0 \equiv \mathcal{M}(R_0)$ and after (33) has been used. Since $R = R_0$ is the critical acceleration boundary, then we immediately have

$$\frac{V_0^2}{R_0} = a_0 = V_\infty^2 \left(\frac{a_0 R_0}{G m_0} \right)$$

from which the scaling relationship follows:

$$\frac{V_\infty^2 R_0}{m_0} = G. \tag{37}$$

Now hypothesize that there exists a class of galaxies for which

$$V_\infty = \sqrt{k}\, V_0 \tag{38}$$

for some global constant k. For this class, (36) gives:

$$\frac{m_0}{R_0^2} = \frac{k\, a_0}{G}, \tag{39}$$

so that, for this hypothetical class of galaxies, Freeman's Law is exactly satisfied.

Eliminating R_0 between (37) and (39), we find:

$$V_\infty^4 = (k a_0 G)\, m_0 \tag{40}$$

so that the baryonic Tully-Fisher relation is also exactly satisfied on the hypothetical galactic class.

11 Summary and Conclusions

The basic work of this paper gives mathematical expression - in the form of a Leibniz-Mach synthesis which pays perfect homage to the conservation laws - to two very old, but simple, ideas:

- the first, mostly associated with the names of Aristotle, Leibniz and Berkeley, can be simply expressed as the idea that *space* is a secondary construct projected out of relationships between *material objects* which, whatever these might be, are primary;
- the second, associated with the names of Leibniz and Mach, is that where *time* is concerned, the most we can ever do is to define the time required for process A to occur in terms of the time required for process B to occur. For example, *I can walk 50 miles between one sunrise and the next.* In effect, the notion of *time passing* is no more and no less than an idealized model for the process of ordered change within a physical system, with one arbitrarily chosen system then providing a standard clock for the generality of timekeeping.

Whilst this Leibniz-Mach synthesis is not limited to any particular domain, in the first instance, it was driven by that which we nominated as the *The Leibniz Question:*

> *Is it possible to conceive a non-trivial global mass distribution, isotropic about every spatial origin, which is in a state of dynamical equilibrium (motion everywhere uniform) and, if so, what are the properties of this distribution?*

The answer to this question, arising as a very special case of the synthesis, is *yes* provided that the mass distribution concerned is fractal, $D = 2$, together with an associated blackbody radiative background. This very special case, which we nominate as $\mathcal{U}_0$ world, corresponds closely to the cosmos which is observed on medium scales and discussed in §2.

Given that the special case of $\mathcal{U}_0$ world corresponds to the observed cosmos on medium scales, we are prompted to consider, briefly, the small scale $\rightarrow$ medium scale boundary for which there is strong circumstantial evidence supporting the idea that this is the MOND critical acceleration boundary (Milgrom (1983a,b)). On the basis of this evidence, we constructed a simple disc-galaxy model and then showed how the Tully-Fisher relation and Freeman's Law arise in a strongly connected way for a very specific class of galaxies - this class being defined by the simple hypothetical property that there exists a class of galaxies for which the rotation velocity at the critical MOND boundary is directly proportional to the asymptotic flat rotation velocity of the rotation curves of the galaxies concerned; that is, the relationship $V_\infty = \sqrt{k}V_0$ (for some global constant k) holds for some subclass of all galaxies.

In summary, we have answered the Leibniz Question, and have arrived at a very specific prediction for the existence of a special class of disc galaxies defined by a simple rotation curve property.

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A A Resolution of the Metric Tensor

The algebra of this section is most easily performed by the change of notation:

$$\Phi \equiv \frac{R^2}{2}, \qquad R^2 \equiv \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2,$$

$$M' \equiv \frac{dM}{d\Phi} \equiv \frac{1}{R}\frac{dM}{dR}, \qquad M'' \equiv \frac{d^2M}{d\Phi^2}, \quad \text{etc.}$$

The general system is given by

$$g_{ab} = \frac{\partial^2 M}{\partial x^a \partial x^b} - \Gamma^k_{ab}\frac{\partial M}{\partial x^k},$$

$$\Gamma^k_{ab} \;\equiv\; \frac{1}{2}g^{kj}\left(\frac{\partial g_{bj}}{\partial x^a} + \frac{\partial g_{ja}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^j}\right),$$

and the first major problem is to express g_{ab} in terms of the mass function, M. The key to resolving this is to note the relationship

$$\frac{\partial^2 M}{\partial x^a \partial x^b} = M'\delta_{ab} + M'' x^a x^b,$$

where $M' \equiv dM/d\Phi$, $M'' \equiv d^2M/d\Phi^2$, $\Phi \equiv R^2/2$, since this immediately suggests the general structure

$$g_{ab} = A\delta_{ab} + Bx^a x^b, \tag{41}$$

for unknown functions, A and B. It is easily found that

$$g^{ab} = \frac{1}{A}\left[\delta_{ab} - \left(\frac{B}{A+2B\Phi}\right)x^a x^b\right]$$

so that, with some effort,

$$\Gamma^k_{ab} = \frac{1}{2A}H_1 - \left(\frac{B}{2A(A+2B\Phi)}\right)H_2$$

where

$$H_1 = A'(x^a\delta_{bk} + x^b\delta_{ak} - x^k\delta_{ab}) + B'x^a x^b x^k + 2B\delta_{ab}x^k$$

and

$$H_2 = A'(2x^a x^b x^k - 2\Phi x^k \delta_{ab}) + 2\Phi B' x^a x^b x^k + 4\Phi B x^k \delta_{ab}.$$

Consequently,

$$g_{ab} = \frac{\partial^2 M}{\partial x^a \partial x^b} - \Gamma^k_{ab}\frac{\partial M}{\partial x^k}$$

$$\equiv \delta_{ab} M' \left(\frac{A + A'\Phi}{A + 2B\Phi} \right) + x^a x^b \left(M'' - M' \left(\frac{A' + B'\Phi}{A + 2B\Phi} \right) \right).$$

Comparison with (41) now leads directly to

$$A = M' \left(\frac{A + A'\Phi}{A + 2B\Phi} \right) = M' \left(\frac{(A\Phi)'}{A + 2B\Phi} \right), \tag{42}$$

$$B = M'' - M' \left(\frac{A' + B'\Phi}{A + 2B\Phi} \right) \tag{43}$$

which is a second order differential equation for the determination of $M(R)$.

The first of the two equations above can be rearranged as

$$B = -\frac{A}{2\Phi} + \frac{M'}{2\Phi} \left(\frac{(A\Phi)'}{A} \right) \tag{44}$$

or as

$$\left(\frac{M'}{A + 2B\Phi} \right) = \frac{A}{(A\Phi)'}, \tag{45}$$

and these expressions can be used to eliminate B in the second equation as follows.

Use of (45) in (43) gives

$$B = M'' - \frac{A}{(A\Phi)'} (A' + B'\Phi) \rightarrow$$

$$(A\Phi)' B + (A\Phi) B' = M'' (A\Phi)' - AA' \rightarrow$$

$$(A\Phi B)' = M'' (A\Phi)' - AA'. \tag{46}$$

But, from (44),

$$A\Phi B = -\frac{1}{2} A^2 + \frac{1}{2} M' (A\Phi)'$$

so that (46) becomes:

$$\left[-\frac{1}{2} A^2 + \frac{1}{2} M' (A\Phi)' \right]' = M'' (A\Phi)' - AA' \rightarrow$$

$$\frac{1}{2} M'' (A\Phi)' + \frac{1}{2} M' (A\Phi)'' = M'' (A\Phi)' \rightarrow$$

$$M' (A\Phi)'' = M'' (A\Phi)' \rightarrow$$

$$(d_0 M') = (A\Phi)' \rightarrow$$

$$d_0\,(M-M_0) = A\Phi \quad \text{where} \quad M_0 \equiv M(0). \tag{47}$$

so that, finally, using (47) and (44), we find for A and B respectively:

$$A \equiv \frac{d_0\,(M-M_0)}{\Phi},$$

$$B \equiv -\frac{A}{2\Phi} + \left(\frac{M'}{2\Phi}\right)\left(\frac{d_0 M'}{A}\right)$$

$$= -\left(\frac{d_0\,(M-M_0)}{2\Phi^2} - \frac{(M-M_0)'\,(M-M_0)'}{2\,(M-M_0)}\right)$$

We can now revert to the notation of the main text $M' \equiv dM/dR$ etc, so that the foregoing results can be expressed as:

$$A \equiv \frac{2d_0\,(M-M_0)}{R^2},$$

$$B \equiv -\frac{1}{2R^2}\left(\frac{4d_0\,(M-M_0)}{R^2} - \frac{(M-M_0)'\,(M-M_0)'}{(M-M_0)}\right).$$

B Conservative Form of Equations of Motion

From the clock constraint equation (22), we have

$$\dot{\mathbf{R}}\cdot\dot{\mathbf{R}} = \frac{c^2}{8\pi\rho_0}A - \frac{B}{A}R^2\dot{R}^2. \tag{48}$$

This suggests defining a potential function as:

$$\mathcal{V} = E_0 - \frac{1}{2}\left(\frac{c^2}{8\pi\rho_0}A - \frac{B}{A}R^2\dot{R}^2\right), \tag{49}$$

for some arbitrary constant E_0, and where A and B are defined at (11). Using the identity

$$\frac{d\mathcal{V}}{dR} \equiv \frac{\partial\mathcal{V}}{\partial R} + \frac{\partial\mathcal{V}}{\partial\dot{R}}\frac{\ddot{R}}{\dot{R}},$$

where we assume that the second term is always defined because the case of circular motions was *explicitly* excluded in the derivation of (17), then we easily obtain

$$\frac{d\mathcal{V}}{dR} = -\frac{c^2}{16\pi\rho_0}A' + \frac{R^2\dot{R}^2}{2A}\left(B' - \frac{A'B}{A}\right) + \frac{B}{A}\left(R\dot{R}^2 + R^2\ddot{R}\right).$$

The above expression leads to

$$2A\frac{d\mathcal{V}}{dR}\hat{\mathbf{R}} = \alpha\,\hat{\mathbf{R}}. \tag{50}$$

where

$$\alpha \equiv \left(-\frac{c^2}{8\pi\rho_0} AA' + B'R^2\dot{R}^2 - \frac{A'B}{A} R^2\dot{R}^2 + 2BR\left(\dot{R}^2 + R\ddot{R}\right) \right)$$

From (48), we have

$$\frac{c^2}{8\pi\rho_0} A = \frac{B}{A} R^2\dot{R}^2 + \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} \tag{51}$$

which, when substituted into (50), gives

$$2A \frac{d\mathcal{V}}{dR} \hat{\mathbf{R}} = \alpha \, \hat{\mathbf{R}}.$$

where

$$\alpha \equiv \left(B'R^2\dot{R}^2 + 2BR\left(\dot{R}^2 + R\ddot{R}\right) - A' \, \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} - 2\frac{A'B}{A} R^2\dot{R}^2 \right)$$

Finally, when used in (24) this gives

$$\ddot{\mathbf{R}} = -\frac{d\mathcal{V}}{dR} \hat{\mathbf{R}}$$

for the result.

C Potential $\mathcal{V}$ purely in terms of R and angular momentum

From the clock constraint equation (21), we have, directly,

$$\dot{R}^2 + R^2\dot{\theta}^2 = \frac{v_0^2}{8\pi\rho_0} A - \frac{B}{A} R^2\dot{R}^2$$

from which we find

$$\left(A + BR^2\right)\dot{R}^2 + AR^2\dot{\theta}^2 = \frac{v_0^2}{8\pi\rho_0} A^2.$$

A small amount of algebra then gives the clock constraint equation as:

$$\dot{R}^2 + R^2\dot{\theta}^2 = \frac{v_0^2}{8\pi\rho_0}\left(\frac{A^2}{A + BR^2}\right) + h^2\left(\frac{B}{A + BR^2}\right)$$

where, for conserved angular momentum, $h \equiv R^2\dot{\theta}$. Consequently, the potential function of (49) can be written as

$$\mathcal{V} = -\frac{1}{2}\left[\frac{v_0^2}{8\pi\rho_0}\left(\frac{A^2}{A + BR^2}\right) + h^2\left(\frac{B}{A + BR^2}\right)\right] \tag{52}$$

From (11), we have

$$A \equiv \frac{2\,d_0\mathcal{M}}{R^2}, \quad B \equiv -\frac{1}{2R^2}\left(\frac{4\,d_0\mathcal{M}}{R^2} - \frac{\mathcal{M}'\mathcal{M}'}{\mathcal{M}}\right). \tag{53}$$

so that

$$A + BR^2 = \frac{\mathcal{M}'\mathcal{M}'}{2\mathcal{M}}.$$

A small amount of algebra now gives, for the expressions in (52)

$$\left(\frac{A^2}{A+BR^2}\right) = \frac{8\,d_0^2\mathcal{M}^3}{R^4\mathcal{M}'\mathcal{M}'}$$

$$\left(\frac{B}{A+BR^2}\right) = -\frac{4\,d_0\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'} + \frac{1}{R^2}$$

so that (52) becomes:

$$\mathcal{V} = -\frac{1}{2}\left[\frac{v_0^2}{8\pi\rho_0}\left(\frac{8\,d_0^2\mathcal{M}^3}{R^4\mathcal{M}'\mathcal{M}'}\right) + h^2\left(-\frac{4\,d_0\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'} + \frac{1}{R^2}\right)\right]$$

and the corresponding form of the clock constraint equation being:

$$\dot{R}^2 + R^2\dot{\theta}^2 = \left[\frac{v_0^2}{8\pi\rho_0}\left(\frac{8\,d_0^2\mathcal{M}^3}{R^4\mathcal{M}'\mathcal{M}'}\right) + h^2\left(-\frac{4\,d_0\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'} + \frac{1}{R^2}\right)\right]$$

$$= 4\,d_0^2v_0^2\left(\frac{\mathcal{M}}{4\pi\rho_0} - \frac{h^2}{d_0v_0^2}\right)\left(\frac{\mathcal{M}^2}{R^4\mathcal{M}'\mathcal{M}'}\right) + \frac{h^2}{R^2} \tag{54}$$

In conventional terms this is equivalent to the energy equation.